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Lectures on the Ricci Flow

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Preface

These notes represent an updated version of a course on Hamilton's Ricci flow that I gave at the University of Warwick in the spring of 2004. I have aimed to give an introduction to the main ideas of the subject, a large proportion of which are due to Hamilton over the period since he introduced the Ricci flow in 1982. The main difference between these notes and others which are available at the time of writing is that I follow the quite different route which is natural in the light of work of Perelman from 2002. It is now understood how to 'blow up' general Ricci flows near their singularities, as one is used to doing in other contexts within geometric analysis. This technique is now central to the subject, and we emphasise it throughout, illustrating it in Chapter 10 by giving a modern proof of Hamilton's theorem that a closed three-dimensional manifold which carries a metric of positive Ricci curvature is a spherical space form.

Aside from the selection of material, there is nothing in these notes which should be considered new. There are quite a few points which have been clarified, and we have given some proofs of well-known facts for which we know of no good reference. The proof we give of Hamilton's theorem does not appear elsewhere in print, but should be clear to experts. The reader will also find some mild reformulations, for example of the curvature pinching results in Chapter 9.

The original lectures were delivered to a mixture of graduate students, postdocs, staff, and even some undergraduates. Generally I assumed that the audience had just completed a first course in differential geometry, and an elementary course in PDE, and were just about to embark on a more advanced course in PDE. I tried to make the lectures accessible to the general mathematician motivated by the applications of the theory to the Poincaré conjecture, and Thurston's geometrisation conjecture (which are discussed briefly in Sections 1.4 and 1.5). This has obviously affected my choice of emphasis. I have suppressed some of the analytical issues, as discussed below, but have compiled a list of relevant Riemannian geometry calculations in Chapter 2.

There are some extremely important aspects of the theory which do not get a mention in these notes. For example, Perelman's $\mathcal{L}$-length, which is a key tool when developing the theory further, and Hamilton's Harnack estimates. There is no discussion of the Kähler-Ricci flow.

I have stopped just short of proving the Hamilton-Ivey pinching result which makes the study of singularities in three-dimensions tractable, although I have covered the necessary techniques to deal with this, and may add an exposition at a later date.

The notes are not completely self-contained. In particular, I state/use the following without giving full proofs:

(i) Existence and uniqueness theory for quasilinear parabolic equations on vector bundles. This is a long story involving rather different techniques to those I focus on in this work. Unfortunately, it is not feasible just to quote theorems from existing sources, and one must learn this theory for oneself;

(ii) Compactness theorems for manifolds and flows. The full proofs of these are long, but a treatment of Ricci flow without using them would be very misleading;

(iii) Parts of Lemma 8.1.8 which involves analysis beyond the level I was assuming. I have given a reference, and intend to give a simple proof in later notes.

These notes are published in the L.M.S. Lecture notes series, in conjunction with Cambridge University Press, and are also available at:

`http://www.maths.warwick.ac.uk/~topping/RFnotes.html`

Readers are invited to send comments and corrections to:

topping@maths.warwick.ac.uk

I would like to thank the audience of the course for making some useful comments, especially Young Choi and Mario Micallef. Thanks also to John Lott for comments on and typographical corrections to a 2005 version of the notes. Parts of the original course benefited from conversations with a number of people, including Klaus Ecker and Miles Simon. Brendan Owens and Gero Friesecke have kindly pointed out some typographical mistakes. Parts of the notes have been prepared whilst visiting the University of Nice, the Albert Einstein Max-Planck Institute in Golm and Free University in Berlin, and I would like to thank these institutions for their hospitality. Finally, I would like to thank Neil Course for preparing all the figures, turning a big chunk of the original course notes into LaTeX, and making some corrections.