1

Introduction

1.1 Ricci flow: what is it, and from where did it come?

Our starting point is a smooth closed (that is, compact and without boundary) manifold $\mathcal{M}$, equipped with a smooth Riemannian metric g. Ricci flow is a means of processing the metric g by allowing it to evolve under the PDE

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) \tag{1.1.1}$$

where $\mathrm{Ric}(g)$ is the Ricci curvature.

In simple situations, the flow can be used to deform g into a metric distinguished by its curvature. For example, if $\mathcal{M}$ is two-dimensional, the Ricci flow, once suitably renormalised, deforms a metric conformally to one of constant curvature, and thus gives a proof of the two-dimensional uniformisation theorem – see Sections 1.4 and 1.5. More generally, the topology of $\mathcal{M}$ may preclude the existence of such distinguished metrics, and the Ricci flow can then be expected to develop a singularity in finite time. Nevertheless, the behaviour of the flow may still serve to tell us much about the topology of the underlying manifold. The present strategy is to stop a flow once a singularity has formed, and then carefully perform 'surgery' on the evolved manifold, excising any singular regions before continuing the flow. Provided we understand the structure of finite time singularities sufficiently well, we may hope to keep track of the change in topology of the manifold under surgery, and reconstruct the topology of the original manifold from a limiting flow, together with the singular regions removed. In these notes, we develop some key elements of the machinery used to analyse singularities, and demonstrate their use by proving Hamilton's theorem that closed three-manifolds which admit a metric of positive Ricci curvature also admit a metric of constant positive sectional curvature.

Of all the possible evolutions for g, one may ask why (1.1.1) has been chosen. As we shall see later, in Section 6.1, one might start by considering a gradient flow for the total scalar curvature of the metric g. This leads to an evolution equation

$$\frac{\partial g}{\partial t} = -\mathrm{Ric} + \frac{R}{2} g,$$

where R is the scalar curvature of g. Unfortunately, this turns out to behave badly from a PDE point of view (see Section 6.1) in that we cannot expect the existence of solutions for arbitrary initial data. Ricci flow can be considered a modification of this idea, first considered by Hamilton [19] in 1982. Only recently, in the work of Perelman [31], has the Ricci flow itself been given a gradient flow formulation (see Chapter 6).

Another justification of (1.1.1) is that from certain viewpoints, $\mathrm{Ric}(g)$ may be considered as a Laplacian of the metric g, making (1.1.1) a variation on the usual heat equation. For example, if for a given metric g we choose harmonic coordinates $\{x^i\}$, then for each fixed pair of indices i and j, we have

$$R_{ij} = -\frac{1}{2}\Delta g_{ij} + \text{lower order terms}$$

where R_{ij} is the corresponding coefficient of the Ricci tensor, and Δ is the Laplace-Beltrami operator which is being applied to the function g_{ij}. Alternatively, one could pick normal coordinates centred at a point p, and then compute that

$$R_{ij} = -\frac{3}{2}\Delta g_{ij}$$

at p, with Δ again the Laplace-Beltrami operator. Beware here that the notation Δg_{ij} would normally refer to the coefficient $(\Delta g)_{ij}$, where Δ is the connection Laplacian (that is, the 'rough' Laplacian) but Δg is necessarily zero since the metric is parallel with respect to the Levi-Civita connection.

1.2 Examples and special solutions

1.2.1 Einstein manifolds

A simple example of a Ricci flow is that starting from a round sphere. This will evolve by shrinking homothetically to a point in finite time.

More generally, if we take a metric g_0 such that

$$\mathrm{Ric}(g_0) = \lambda g_0$$

for some constant $\lambda \in \mathbb{R}$ (these metrics are known as Einstein metrics) then a solution $g(t)$ of (1.1.1) with $g(0) = g_0$ is given by

$$g(t) = (1 - 2\lambda t)g_0.$$

(It is worth pointing out here that the Ricci tensor is invariant under uniform scalings of the metric.) In particular, for the round 'unit' sphere (S^n, g_0), we have $\mathrm{Ric}(g_0) = (n-1)g_0$, so the evolution is $g(t) = (1 - 2(n-1)t)g_0$ and the sphere collapses to a point at time $T = \frac{1}{2(n-1)}$.

An alternative example of this type would be if g_0 were a hyperbolic metric – that is, of constant sectional curvature -1. In this case $\mathrm{Ric}(g_0) = -(n-1)g_0$, the evolution is $g(t) = (1 + 2(n-1)t)g_0$ and the manifold *expands* homothetically for all time.

1.2.2 Ricci solitons

There is a more general notion of self-similar solution than the uniformly shrinking or expanding solutions of the previous section. We consider these 'Ricci solitons' without the assumption that $\mathcal{M}$ is compact. To understand such solutions, we must consider the idea of modifying a flow by a family of diffeomorphisms. Let $X(t)$ be a time dependent family of smooth vector fields on $\mathcal{M}$, generating a family of diffeomorphisms ψ_t. In other words, for a smooth $f : \mathcal{M} \to \mathbb{R}$, we have

$$X(\psi_t(y), t)f = \frac{\partial f \circ \psi_t}{\partial t}(y). \tag{1.2.1}$$

Of course, we could start with a family of diffeomorphisms ψ_t and define $X(t)$ from it, using (1.2.1).

Next, let $\sigma(t)$ be a smooth function of time.

Proposition 1.2.1. *Defining*

$$\hat{g}(t) = \sigma(t)\psi_t^*(g(t)), \tag{1.2.2}$$

we have

$$\frac{\partial \hat{g}}{\partial t} = \sigma'(t)\psi_t^*(g) + \sigma(t)\psi_t^*\left(\frac{\partial g}{\partial t}\right) + \sigma(t)\psi_t^*(\mathcal{L}_X g). \tag{1.2.3}$$

This follows from the definition of the Lie derivative. (It may help you to write $\psi_t^*(g(t)) = \psi_t^*(g(t) - g(s)) + \psi_t^*(g(s))$ and differentiate at $t = s$.) As a consequence of this proposition, if we have a metric g_0, a vector field Y and $\lambda \in \mathbb{R}$ (all independent of time) such that

$$-2\mathrm{Ric}(g_0) = \mathcal{L}_Y g_0 - 2\lambda g_0, \tag{1.2.4}$$

then after setting $g(t) = g_0$ and $\sigma(t) := 1 - 2\lambda t$, if we can integrate the t-dependent vector field $X(t) := \frac{1}{\sigma(t)}Y$, to give a family of diffeomorphisms ψ_t with ψ_0 the identity, then $\hat{g}$ defined by (1.2.2) is a Ricci flow with $\hat{g}(0) = g_0$:

$$\begin{aligned}\frac{\partial \hat{g}}{\partial t} = \sigma'(t)\psi_t^*(g_0) + \sigma(t)\psi_t^*(\mathcal{L}_X g_0) &= \psi_t^*(-2\lambda g_0 + \mathcal{L}_Y g_0)\\ &= \psi_t^*(-2\mathrm{Ric}(g_0))\\ &= -2\mathrm{Ric}(\psi_t^* g_0)\\ &= -2\mathrm{Ric}(\hat{g}).\end{aligned}$$

(Note again that the Ricci tensor is invariant under uniform scalings of the metric.)

Definition 1.2.2. Such a flow is called a steady, expanding or shrinking 'Ricci soliton' depending on whether $\lambda = 0$, $\lambda < 0$ or $\lambda > 0$ respectively.

Conversely, given any Ricci flow $\hat{g}(t)$ of the form (1.2.2) for some $\sigma(t)$, ψ_t, and $g(t) = g_0$, we may differentiate (1.2.2) at $t = 0$ (assuming smoothness) to show that g_0 is a solution of (1.2.4) for appropriate Y and λ. If we are in a situation where we can be sure of uniqueness of solutions (see Theorem 5.2.2 for one such situation) then our $\hat{g}(t)$ must be the Ricci soliton we have recently constructed.[1]

Definition 1.2.3. A Ricci soliton whose vector field Y can be written as the gradient of some function $f : \mathcal{M} \to \mathbb{R}$ is known as a 'gradient Ricci soliton.'

In this case, we may compute that $\mathcal{L}_Y g_0 = 2\mathrm{Hess}_{g_0}(f)$ (we will review this fact in (2.3.9) below) and so by (1.2.4), f satisfies

$$\mathrm{Hess}_{g_0}(f) + \mathrm{Ric}(g_0) = \lambda g_0. \tag{1.2.5}$$

Hamilton's cigar soliton (a.k.a. Witten's black hole)
Let $\mathcal{M} = \mathbb{R}^2$, and write $g_0 = \rho^2(dx^2 + dy^2)$, using the convention $dx^2 = dx \otimes dx$. The formula for the Gauss curvature is

$$K = -\frac{1}{\rho^2}\Delta \ln \rho,$$

where this time we are writing $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and the Ricci curvature can be written in terms of the Gauss curvature as $\mathrm{Ric}(g_0) = K g_0$. If now we set

[1] One should beware that uniqueness may fail in general. For example, one can have two distinct (smooth) Ricci flows on a time interval $[0, T]$ starting at the same (incomplete) g_0, even if we ask that each is a soliton for $t \in (0, T]$. (See [40].)

$\rho^2 = \frac{1}{1+x^2+y^2}$, then we find that $K = \frac{2}{1+x^2+y^2}$, that is,

$$\mathrm{Ric}(g_0) = \frac{2}{1+x^2+y^2} g_0. \tag{1.2.6}$$

Meanwhile, if we define Y to be the radial vector field $Y := -2(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$, then one can calculate that

$$\mathcal{L}_Y g_0 = -\frac{4}{1+x^2+y^2} g_0.$$

Therefore by (1.2.4), g_0 flows as a steady ($\lambda = 0$) Ricci soliton.

It is illuminating to write g_0 in terms of the geodesic distance from the origin s, and the polar angle θ to give

$$g_0 = ds^2 + \tanh^2 s\, d\theta^2.$$

This shows that the cigar opens at infinity like a cylinder – and therefore looks like a cigar! It is useful to know the curvature in these coordinates:

$$K = \frac{2}{\cosh^2 s}.$$

Finally, note that the cigar is also a *gradient* soliton since Y is radial. Indeed, we may take $f = -2 \ln \cosh s$.

The cigar is one of many Ricci solitons which can be written down explicitly. However, it has been distinguished historically as part of one of the possible limits one could find when making an appropriate rescaling (or "blow-up") of *three*-dimensional Ricci flows near finite-time singularities. Only recently, with work of Perelman, has this possibility been ruled out. The blowing-up of flows near singularities will be discussed in Sections 7.3 and 8.5.

The Bryant soliton

There is a similar rotationally symmetric steady gradient soliton for $\mathbb{R}^3$, found by Bryant. Instead of opening like a cylinder at infinity (as is the case for the cigar soliton) the Bryant soliton opens asymptotically like a paraboloid. It has positive sectional curvature.

The Gaussian soliton

One might consider the stationary (that is, time independent) flow of the standard flat metric on $\mathbb{R}^n$ to be quite boring. However, it may later be useful to consider it as a gradient Ricci soliton in more than one way. First, one may take $\lambda = 0$ and $Y \equiv 0$, and see it as a steady soliton. However, for *any* $\lambda \in \mathbb{R}$, one may set $f(x) = \frac{\lambda}{2}|x|^2$, to see the flow as either an expanding or shrinking soliton depending on the sign of λ. (Note that $\psi_t(x) = (1 + \lambda t)x$, and $\mathcal{L}_Y g = 2\lambda g$.)

1.2.3 Parabolic rescaling of Ricci flows

Suppose that $g(t)$ is a Ricci flow for $t \in [0, T]$. (Implicit in this statement here, and throughout these notes, is that $g(t)$ is a smooth family of smooth metrics – smooth all the way to $t = 0$ and $t = T$ – which satisfies (1.1.1).) Given a scaling factor $\lambda > 0$, if one defines a new flow by scaling time by λ and distances by $\lambda^{\frac{1}{2}}$, that is one defines

$$\hat{g}(x, t) = \lambda g(x, t/\lambda), \tag{1.2.7}$$

for $t \in [0, \lambda T]$, then

$$\frac{\partial \hat{g}}{\partial t}(x, t) = \frac{\partial g}{\partial t}(x, t/\lambda) = -2\text{Ric}(g(t/\lambda))(x) = -2\text{Ric}(\hat{g}(t))(x), \tag{1.2.8}$$

and so $\hat{g}$ is also a Ricci flow. Under this scaling, the Ricci tensor is invariant, as we have just used again, but sectional curvatures and the scalar curvature are scaled by a factor λ^{-1}; for example,

$$R(\hat{g}(x, t)) = \lambda^{-1} R(g(x, t/\lambda)). \tag{1.2.9}$$

The connection also remains invariant.

The main use of this rescaling will be to analyse Ricci flows which develop singularities. We will see in Section 5.3 that such flows have curvature which blows up (that is, tends to infinity in magnitude) and much of our effort during these notes will be to develop a way of rescaling the flow where the curvature is becoming large in such a way that we can pass to a limit which will be a new Ricci flow encoding some of the information contained in the singularity. This is a very successful strategy in many branches of geometric analysis. Blow-up limits in other problems include tangent cones of minimal surfaces and bubbles in the harmonic map flow.

1.3 Getting a feel for Ricci flow

We have already seen some explicit, rigorous examples of Ricci flows, but it is important to get a feel for how we expect more general Ricci flows, with various shapes and dimensions, to evolve. We approach this from a purely heuristic point of view.

1.3.1 Two dimensions

In two dimensions, we know that the Ricci curvature can be written in terms of the Gauss curvature K as $\text{Ric}(g) = Kg$. Working directly from the equation

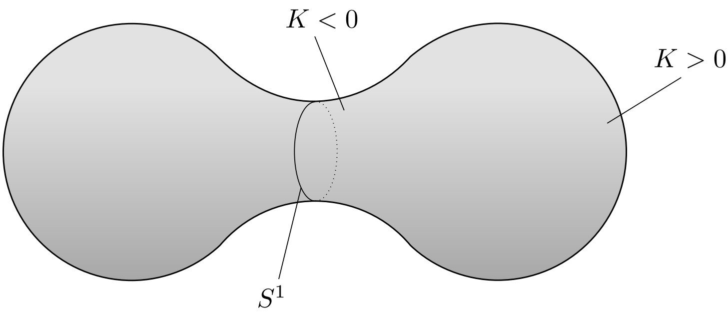


Figure 1.1 2-sphere

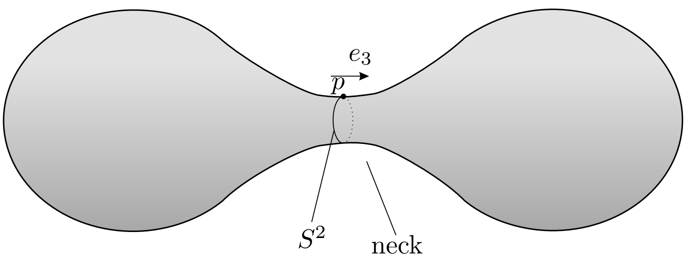


Figure 1.2 3-sphere

(1.1.1), we then see that regions in which $K < 0$ tend to expand, and regions where $K > 0$ tend to shrink.

By inspection of Figure 1.1, one might then guess that the Ricci flow tends to make a 2-sphere “rounder”. This is indeed the case, and there is an excellent theory which shows that the Ricci flow on any closed surface tends to make the Gauss curvature constant, after renormalisation. See the book of Chow and Knopf [7] for more information about this specific dimension.

1.3.2 Three dimensions

The neck pinch

The three-dimensional case is more complicated, but we can gain useful intuition by considering the flow of an analogous three-sphere.

Now the cross-sectional sphere is an S^2 (rather than an S^1 as it was before) as indicated in Figure 1.2, and it has its own positive curvature. Let e_1, e_2, e_3 be orthonormal vectors at the point p in Figure 1.2, with e_3 perpendicular to the indicated cross-sectional S^2. Then the sectional curvatures $K_{e_1 \wedge e_3}$ and $K_{e_2 \wedge e_3}$ of the ‘planes’ $e_1 \wedge e_3$ and $e_2 \wedge e_3$ are slightly negative (c.f. Figure 1.1) but $K_{e_1 \wedge e_2}$

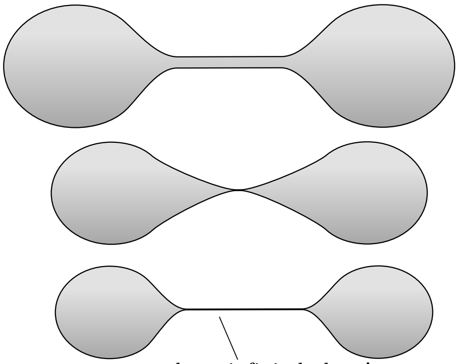


Figure 1.3 Neck Pinch

is very positive. Therefore

$$\begin{aligned}\mathrm{Ric}(e_1, e_1) &= K_{e_1\wedge e_2} + K_{e_1\wedge e_3} = \text{very positive}\\ \mathrm{Ric}(e_2, e_2) &= K_{e_2\wedge e_1} + K_{e_2\wedge e_3} = \text{very positive}\\ \mathrm{Ric}(e_3, e_3) &= K_{e_3\wedge e_1} + K_{e_3\wedge e_2} = \text{slightly negative}\end{aligned}$$

This information indicates how the manifold begins to evolve under the Ricci flow equation (1.1.1). We expect that distances shrink rapidly in the e_1 and e_2 directions, but expand slowly in the e_3 direction. Thus, the metric wants to quickly contract the cross-sectional S^2 indicated in Figure 1.2, whilst slowly stretching the neck. At later times, we expect to see something like the picture in Figure 1.3(i) and perhaps eventually 1.3(ii) or maybe even 1.3(iii).

Only recently have theorems been available which rigorously establish that something like this behaviour does sometimes happen. For more information, see [1] and [37].

It is important to get some understanding of the exact structure of this process. Singularities are typically analysed by blowing up: Where the curvature is large, we magnify – that is, rescale or 'blow-up' – so that the curvature is no longer large, as in Figure 1.4. (Recall the discussion of rescaling in Section 1.2.3.) We will work quite hard to make this blowing-up process precise and rigorous, with the discussion centred on Sections 7.3 and 8.5.

In this particular instance, the blow-up looks like a part of the cylinder $S^2 \times \mathbb{R}$ (a 'neck') and the most advanced theory in three-dimensions tells us that in some sense this is typical. See [31] for more information.

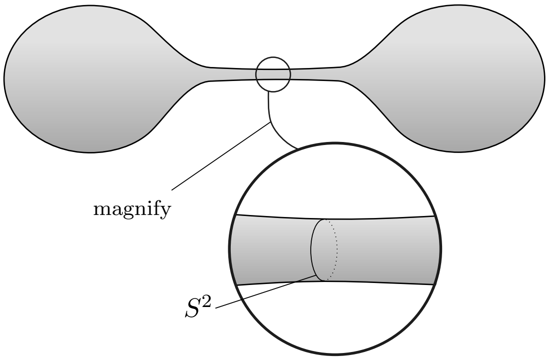
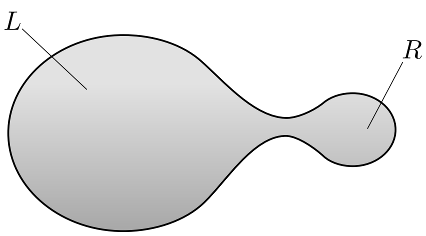


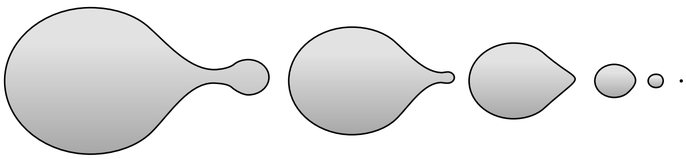
Figure 1.4 Blowing up

The degenerate neck pinch

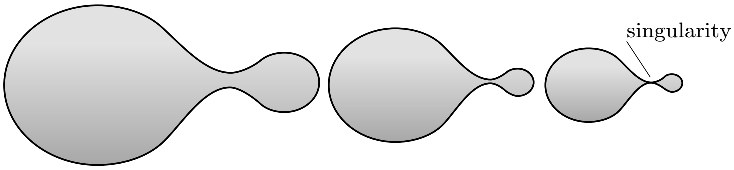
One possible blow-up, the existence of which we shall not try to make rigorous, is the degenerate neck pinch. Consider the flow of a similar, but asymmetrical three-sphere of the following form:



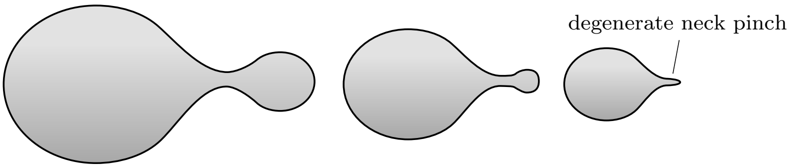
If the part R is small, then the flow should look like:



and the manifold should look asymptotically like a small sphere. Meanwhile, if the part R is large, then the flow should look like:



Somewhere in between (when R is of just the right size), we should have:



If we were to blow-up this singularity, then we should get something looking like the Bryant soliton:

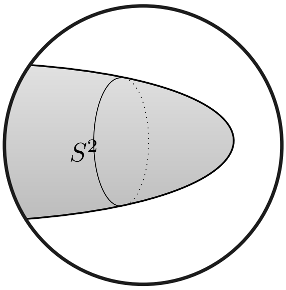
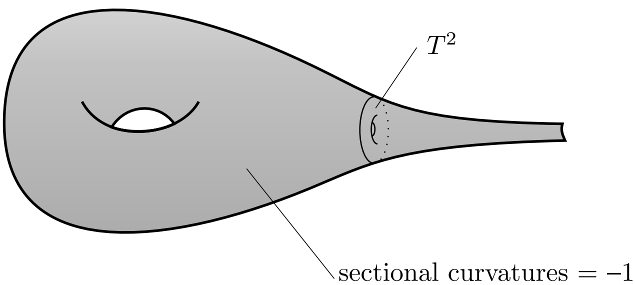


Figure 1.5 Magnified degenerate neck pinch

Infinite time behaviour

A Ricci flow need not converge as $t \to \infty$. In our discussion of Einstein manifolds (Section 1.2.1) we saw that a hyperbolic manifold continues to expand forever, and in Section 1.2.2 we wrote down examples such as the cigar soliton which evolve in a more complicated way. Even if we renormalise our flow, or adjust it by a time-dependent diffeomorphism, we cannot expect convergence, and the behaviour of the flow could be quite complicated. We now give a rough sketch of one flow we should expect to see at 'infinite time'.

Imagine a hyperbolic three-manifold with a toroidal end.



This would expand homothetically under the Ricci flow, as we discussed in Section 1.2.1. Now paste two such pieces together, breaking the hyperbolicity