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Synthetic Differential Geometry

2^{nd} edition

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Contents

Preface to the Second Edition (2006)

The First Edition (1981) of "Synthetic Differential Geometry" has been out of print since the early 1990s. I felt that there was still a need for the book, even though other accounts of the subject have in the meantime come into existence.

Therefore I decided to bring out this Second Edition. It is a compromise between a mere photographic reproduction of the First Edition, and a complete rewriting of it. I realized that a rewriting would quickly lead to an almost new book. I do indeed intend to write a new book, but prefer it to be a *sequel* to the old one, rather than a rewriting of it.

For the same reason, I have refrained from attempting an account of all the developments that have taken place since the First Edition; only very minimal and incomplete pointers to the newer literature (1981–2006) have been included as "Notes 2006" at the end of each of the Parts of the book.

Most of the basic notions of synthetic differential geometry were already in the 1981 book; the main exception being the general notion of "strong infinitesimal linearity" or "microlinearity", which came into being just too late to be included. A small Appendix D on this notion is therefore added.

Otherwise, the present edition is a re-typing of the old one, with only minor corrections, where necessary. In particular, the numberings of Parts, equations, etc. are unchanged. The bibliography consists of two parts: the first one (entries [1] to [81]) is identical to the bibliography from the 1981 edition, the second one (from entry [82] onwards) contains later literature, as referred to in the end-notes (so it is not meant to be complete; I hope in a possible forthcoming Second Book to be able to survey the field more completely).

Besides the thanks that are expressed in the Preface to the 1981 edition (as reprinted following), I would like to express thanks to Prof. Andrée Charles Ehresmann for her tireless work in running the journal *Cahiers de Topologie et Géométrie Différentielle Catégoriques*. This journal has for a couple of decades been essential for the exchange and dissemination of knowledge about Synthetic Differential Geometry (as well as of many other topics in Mathematics).

I would like to thank Eduardo Dubuc, Joachim Kock, Bill Lawvere, and Gonzalo Reyes for useful comments on this Second Edition.

I also want to thank the staff of Cambridge University Press for technical assistance in the preparation of this Second Edition. Most diagrams were drawn using Paul Taylor's "Diagrams" package.

Preface to the First Edition (1981)

The aim of the present book is to describe a foundation for synthetic reasoning in differential geometry. We hope that such a foundational treatise will put the reader in a position where he, in his study of differential geometry, can utilize the synthetic method freely and rigorously, and that it will give him notions and language by which such study can be communicated.

That such notions and language is something that till recently seems to have existed only in an inadequate way is borne out by the following statement of Sophus Lie, in the preface to one of his fundamental articles:

> *"The reason why I have postponed for so long these investigations, which are basic to my other work in this field, is essentially the following. I found these theories originally by synthetic considerations. But I soon realized that, as expedient [zweckmässig] the synthetic method is for discovery, as difficult it is to give a clear exposition on synthetic investigations, which deal with objects that till now have almost exclusively been considered analytically. After long vacillations, I have decided to use a half synthetic, half analytic form. I hope my work will serve to bring justification to the synthetic method besides the analytical one."*
>
> (Allgemeine Theorie der partiellen Differentialgleichungen erster Ordnung, Math. Ann. 9 (1876).)

What is meant by "synthetic" reasoning? Of course, we do not know exactly what Lie meant, but the following is the way we would describe it: It deals with space forms in terms of their structure, i.e. the basic geometric and conceptual constructions that can be performed on them. Roughly, these constructions are the morphisms which constitute the base category in terms of which we work; the space forms themselves being objects of it.

This category is *cartesian closed*, since, whenever we have formed ideas of "spaces" A and B, we can form the idea of B^A, the "space" of all functions from A to B.

The category theoretic viewpoint prevents the identification of A and B with point sets (and hence also prevents the formation of "random" maps from A to B). This is an old tradition in synthetic geometry, where one, for instance, distinguishes between a "*line*" and the "*range* of points on it" (cf. e.g. Coxeter [8] p. 20).

What categories in the "Bourbakian" universe of mathematics are mathematical models of this intuitively conceived geometric category? The answer is: many of the "gros toposes" considered since the early 1960s by Grothendieck and others, – the simplest example being the category of functors from commutative rings to sets. We deal with these topos theoretic examples in Part III of the book. We do not begin with them, but rather with the axiomatic development of differential geometry on a synthetic basis (Part I), as well as a method of interpreting such development in cartesian closed categories (Part II). We chose this ordering because we want to stress that the *axioms* are intended to reflect some true properties of the geometric and physical reality; the *models* in Part III are only servants providing consistency proofs and inspiration for new true axioms or theorems. We present in particular some models $\mathcal{E}$ which contain the category of smooth manifolds as a full subcategory in such a way that "analytic" differential geometry for these corresponds exactly to "synthetic" differential geometry in $\mathcal{E}$.

Most of Part I, as well as several of the papers in the bibliography which go deeper into actual geometric matters with synthetic methods, are written in the "naive" style.[1] By this, we mean that all notions, constructions, and proofs involved are presented *as if* the base category were the category of sets; in particular all constructions on the objects involved are described in terms of "*elements*" of them. However, it is necessary and possible to be able to understand this naive writing as referring to cartesian closed categories. It is *necessary* because the basic axioms of synthetic differential geometry have no models in the category of sets (cf. I §1); and it is possible: this is what Part II is about. The method is that we have to understand by an *element* b of an object B a *generalized* element, that is, a map $b : X \to B$, where X is an arbitrary object, called the *stage of definition*, or the *domain of variation* of the element b.

Elements "defined at different stages" have a long tradition in geometry. In fact, a special case of it is when the geometers say: A circle has no *real* points at infinity, but there are two *imaginary* points at infinity such that every circle passes through them. Here $\mathbb{R}$ and $\mathbb{C}$ are two different *stages* of mathematical knowledge, and something that does not yet exist at stage $\mathbb{R}$ may come into existence at the "later" or "deeper" stage $\mathbb{C}$. – More important for the developments here are passage from stage $\mathbb{R}$ to stage $\mathbb{R}[\epsilon]$, the "ring of dual numbers over $\mathbb{R}$":

$$\mathbb{R}[\epsilon] = \mathbb{R}[x]/(x^2).$$

It is true, and will be apparent in Part III, that the notion of elements defined at different stages does correspond to this classical notion of elements defined relative to different commutative rings, like $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}[\epsilon]$, cf. the remarks at the end of III §1.

When thinking in terms of physics (of which geometry of space forms is a special case), the reason for the name "domain of variation" (instead of "stage of definition") becomes clear: for a non-atomistic point of view, a body B is not described just in terms of its "atoms" $b \in B$, that is, maps $\mathbf{1} \to B$, but in terms of "particles" of varying size X, or in terms of motions that take place in B and are parametrized by a temporal extent X; both of these situations being described by maps $X \to B$ for suitable domain of variation X.

The exercises at the end of each paragraph are intended to serve as a further source of information, and if one does not want to solve them, one might read them.

Historical remarks and credits concerning the main text are collected at the end of the book. If a specific result is not credited to anybody, it does not necessarily mean that I claim credit for it. Many things developed during discussions between Lawvere, Wraith, myself, Reyes, Joyal, Dubuc, Coste, Coste-Roy, Bkouche, Veit, Penon, and others. Personally, I want to acknowledge also stimulating questions, comments, and encouragement from Dana Scott, J. Bénabou, P. Johnstone, and from my audiences in Milano, Montréal, Paris, Zaragoza, Buffalo, Oxford, and, in particular, Aarhus. I want also to thank Henry Thomsen for valuable comments to the early drafts of the book.

The Danish Natural Science Research Council has on several occasions made it possible to gather some of the above-mentioned mathematicians for work sessions in Aarhus. This has been vital to the progress of the subject treated here, and I want to express my thanks.

Warm thanks also to the secretaries at Matematisk Institut, Aarhus, for their friendly help, and in particular, to Else Yndgaard for her expert typing of this book.[2]

Finally, I want to thank my family for all their support, and for their patience with me and the above-mentioned friends and colleagues.

Notes 2006

[1]Lavendhomme [131] uses the word ‘naive’ synonymously with ‘synthetic’. Modelled after Synthetic Differential Geometry, the idea of a Synthetic Domain Theory came into being in the late 1980s, cf. [102]. A study of topos models for both these “synthetic” theories is promised for Johnstone’s forthcoming “Elephant” Vol. III, [104].

[2]This refers to the First Edition, 1981; the present Second Edition was scanned/typed by myself.