SATURATED EXTENSIONS, THE ATTRACTORS METHOD AND HEREDITARILY JAMES TREE SPACES

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Dedicated to the memory of R.C. James

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0. Introduction

The purpose of the present work is to provide examples of HI Banach spaces with no reflexive subspace and study their structure. As is well known W.T. Gowers [G1] has constructed a Banach space $\mathfrak{X}_{gt}$ with a boundedly complete basis $(e_n)_n$, not containing ℓ_1, and such that all of its infinite dimensional subspaces have non separable dual. We shall refer to this space as the Gowers Tree space. The predual $(\mathfrak{X}_{gt})_*$, namely the space generated by the biorthogonal of the basis, also has the property that it does not contain c_0 or a reflexive subspace. It remains unknown whether $\mathfrak{X}_{gt}$ is HI and moreover the structure of $\mathcal{L}(\mathfrak{X}_{gt})$ is unclear. Notice that Gowers dichotomy [G2] yields that $\mathfrak{X}_{gt}$ and $(\mathfrak{X}_{gt})_*$ contain HI subspaces. The structure of $\mathfrak{X}^*_{gt}$ also remains unclear. The main obstacle for understanding the structure of $\mathfrak{X}_{gt}$ or $\mathcal{L}(\mathfrak{X}_{gt})$ is the use of

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a probabilistic argument for establishing the existence of vectors with certain properties.

Our approach in constructing HI spaces with no reflexive subspace, is different from Gowers' one. In particular we avoid the use of any probabilistic argument and thus we are able to control the structure of the spaces as well as the structure of the spaces of bounded linear operators acting on them. Moreover we are able to provide examples of spaces X exhibiting a vast difference between the structures of X and X^*.

The following are the highlight of our results:

- There exists a HI Banach space X with a shrinking basis and with no reflexive subspace. Moreover every $T : X \to X$ is of the form $\lambda I + W$ with W weakly compact (and hence strictly singular).

The absence of reflexive subspaces in X in conjunction with the property that every strictly singular operator is weakly compact is evidence supporting the existence of Banach spaces such that every non Fredholm operator is compact.

- The dual X^* of the previous X is HI and reflexively saturated and the dual of every subspace Y of X is also HI.

This shows a strong divergence between the structure of X and X^*. We recall that in [AT2] a reflexive HI space X is constructed whose dual X^* is unconditionally saturated. The analogue of this in the present setting is the following one:

- There exists a HI Banach space Y with a shrinking basis and with no reflexive subspace, such that the dual space Y^* is reflexive and unconditionally saturated.

The definition of the space Y requires an adaptation of the methods of [AT2] within the present framework of building spaces with no reflexive subspace.

- There exists a partition of the basis $(e_n)_n$ of the previous X into two sets $(e_n)_{n\in L_1}$, $(e_n)_{n\in L_2}$ such that setting $X_{L_1} = \overline{\text{span}}\{e_n : n \in L_1\}$, $X_{L_2} = \overline{\text{span}}\{e_n : n \in L_2\}$, both $X_{L_1}^*$, $X_{L_2}^*$ are HI with no reflexive subspace.

The pairs $X_{L_i}, X_{L_i}^*$ for $i = 1, 2$ share similar properties with the pair $(\mathfrak{X}_{gt})_*$ and $\mathfrak{X}_{gt}$.

- The space X^{**} is non separable and every w^*-closed subspace of X^{**}, is either isomorphic to ℓ_2 or is non-separable and contains ℓ_2. Therefore every quotient of X^* has a further quotient isomorphic to ℓ_2. Moreover X^{**}/X is isomorphic to $\ell_2(\Gamma)$.

It seems also possible that $\mathfrak{X}_{gt}^*$ satisfies a similar to the above property although this is not easily shown. Further X^* is the first example of a HI space with the following property: X^*/Y is HI whenever Y is w^*-closed (this is equivalent to say that for every subspace Z of X, Z^* is HI) and also every quotient of X^* has a further quotient which is isomorphic to ℓ_2.

- There exists a non separable HI Banach space Z not containing a reflexive subspace and such that every bounded linear operator T :

$Z \to Z$ is of the form $T = \lambda I + W$ with W a weakly compact (hence strictly singular) operator with separable range.

This is an extreme construction resulting from a variation of the methods used in the construction of the space X involved in the previous results. The fact of the matter is that these methods are not stable. Thus some minor changes in the initial data could produce spaces with entirely different structure. Notice that the space Z is of the form Y^{**} with Y and Y^* sharing similar properties with the pair X, X^* appearing in the previous statements.

We shall proceed to a more detailed presentation of the results of the paper and also of the methods used for constructing the spaces, which are interesting on their own. We have divided the rest of the introduction in three subsections. The first concerns the structure of Banach spaces not containing ℓ_1, c_0 or reflexive subspace. The second is devoted to saturated extensions and in the third we explain the method of attractors which permits the construction of dual pairs X, X^* with strongly divergent structure.

0.1. **Hereditarily James Tree spaces.** Separable spaces like Gowers Tree space undoubtedly have peculiar structure. Roughly speaking, in every subspace one can find a structure similar to the James tree basis. Next we shall attempt to be more precise. Thus we shall define the Hereditarily James Tree spaces, making more transparent their structure. We begin by recalling some of the fundamental characteristics of James' paradigm.

In the sequel we shall denote by $(\mathcal{D}, \prec)$ the dyadic tree and by $[\mathcal{D}]$, the set of all branches (or the body) of $\mathcal{D}$. As usual we would consider that the nodes of $\mathcal{D}$ consist of finite sequences of 0's and 1's and $a \prec b$ iff a is an initial part of b. The lexicographic order of $\mathcal{D}$, denoted by $\prec_{lex}$ defines a well ordering which is consistent with the tree order (i.e. $a \prec b$ yields that $a \prec_{lex} b$).

The space JT.

The James Tree space JT ([J]) is defined as the completion of $(c_{00}(\mathcal{D}), \|\cdot\|_{JT})$ where for $x \in c_{00}(\mathcal{D})$, $\|x\|_{JT}$ is defined as follows:

$$\|x\|_{JT} = \sup\Big\{\Big(\sum_{i=1}^{n}\Big(\sum_{n\in s_i} x(n)\Big)^2\Big)^{1/2} : \ (s_i)_{i=1}^n \text{ pairwise disjoint segments } \Big\}.$$

The main properties of the space JT, is that does not contain ℓ_1 and has nonseparable dual.

Next, we list some properties of JT related to our consideration.

- The Hamel basis $(e_a)_{a\in\mathcal{D}}$ of $c_{00}(\mathcal{D})$ ordered with the lexicographic order defines a (conditional) boundedly complete basis of JT.
- For every branch b in $[\mathcal{D}]$, $b = (a_1 \prec a_2 \prec \cdots \prec a_n \cdots)$ the sequence $(e_{a_n})_n$ is non trivial weak Cauchy and moreover $b^* = w^* - \sum_{n=1}^{\infty} e^*_{a_n}$ defines a norm one functional in JT^*.
- The biorthogonal functionals of the basis $(e^*_a)_{a\in\mathcal{D}}$ generate the predual JT_* of JT and they satisfy the following property.

  For every segment s of $\mathcal{D}$ setting $s^* = \sum_{a\in s} e^*_a$ we have that $\|s^*\| = 1$.

It is worth pointing out an alternative definition of the norm of JT. Thus we consider the following subset of $c_{00}(\mathcal{D})$,

$$G_{JT} = \{ \sum_{i=1}^{n} \lambda_i s_i^* : (s_i)_{i=1}^n \text{ are disjoint finite segments and } \sum_{i=1}^{n} \lambda_i^2 \leq 1 \}$$

Here s_i^* are defined as before. It is an easy exercise to see that the norm induced by the set G_{JT} on $c_{00}(\mathcal{D})$ coincides with the norm of JT.

The James Tree properties.

Let X be a space with a Schauder basis $(e_n)_n$. A block subspace Y of X has the boundedly complete (shrinking) James tree property if there exists a seminormalized block (in the lexicographical order $\prec_{lex}$ of $\mathcal{D}$) sequence $(y_a)_{a\in\mathcal{D}}$ in Y and a $c > 0$ such that the following holds.

(1) **(boundedly complete)** There exists a bounded family $(b^*)_{b\in[\mathcal{D}]}$ in X^*, such that for each $b \in [\mathcal{D}]$, $b = (a_1, a_2, \ldots, a_n, \ldots)$ the sequence $(y_{a_n})_n$ is non trivial weakly Cauchy with $\lim b^*(y_{a_n}) > c$ and $\lim b_1^*(y_{a_n}) = 0$ for all $b_1 \neq b$.

(2) **(shrinking)** For all finite segments s of $\mathcal{D}$, $\| \sum_{a\in s} y_a \| \leq c$.

Let's observe that $(e_a)_{a\in\mathcal{D}}$ in JT satisfies the boundedly complete James Tree property while $(e_a^*)_{a\in\mathcal{D}}$ in JT_* satisfies the shrinking one. Also, if the initial space X has a boundedly complete basis only the boundedly complete James Tree property could occur. A similar result holds if X has a shrinking basis. Finally if Y has the boundedly complete James Tree property, then Y^* is non separable and if X has a shrinking basis and Y has the (shrinking) James Tree property, then Y^{**} is non separable.

For simplicity, in the sequel we shall consider that the initial space X has either a boundedly complete or a shrinking basis. Thus if a block subspace has the James Tree property, then it will be determined as either boundedly complete or shrinking according to the corresponding property of the initial basis.

Definition 0.1. Let X be a Banach space with a Schauder basis.

(a) A family $\mathcal{L}$ of block subspaces of X has the James Tree property, provided every Y in $\mathcal{L}$ has that property.

(b) The space X is said to be Hereditarily James Tree (HJT) if it does not contain c_0, ℓ_1 and every block subspace Y of X, has the James Tree property.

It follows from Gowers' construction that the Gowers Tree space $\mathfrak{X}_{gt}$, and its predual $(\mathfrak{X}_{gt})_*$ are HJT spaces.

One of the results of the present work is that HJT property is not preserved under duality. Namely, there exists a HJT space X with a shrinking basis, such that X^* is reflexively (even unconditionally) saturated. However, in the same example there exists a subspace Y of X with Y^* also an HJT space.

One of the basic ingredients in our approach to building HJT spaces is the following space:

Proposition 0.2. There exists a space $JT_{\mathcal{F}_2}$ with a boundedly complete basis $(e_n)_n$ such that the following hold:

(i) The space $JT_{\mathcal{F}_2}$ is ℓ_2 saturated.
(ii) The basis $(e_n)_n$ is normalized weakly null and for every $M \in [\mathbb{N}]$ the subspace $X_M = \overline{\text{span}}\{e_n : n \in M\}$ has the James tree property.

It is clear that none subsequence $(e_n)_{n\in M}$ is unconditional. Thus the basis of $JT_{\mathcal{F}_2}$ shares similar properties with the classical Maurey Rosenthal example [MR]. We shall return to this space in the sequel explaining more about its structure and its difference from Gowers' space.

Codings and tree structures. As is well known, every attempt to impose tight (or conditional) structure in some Banach space, requires the definition of the conditional elements which in turn results from the existence of special sequences defined with the use of a coding. What is less well known is that the codings induce a tree structure in the special sequences. As we shall explain shortly, the James tree structure in the subspaces of HJT spaces, like $\mathfrak{X}_{gt}$, $(\mathfrak{X}_{gt})_*$ or the spaces presented in this paper, are directly related to codings.

Let's start with a general definition of a coding, and the obtained special sequences. Consider a collection $(F_j)_j$ with each F_j a countable family of elements of $c_{00}(\mathbb{N})$. To make more transparent the meaning of our definitions, let's assume that each $F_j = \{\frac{1}{m_j^2}\sum_{k\in F} e_k^* : \ F \subset \mathbb{N},\ \#F \leq n_j\}$ where $(m_j), (n_j)$ are appropriate fast increasing sequences of natural numbers. Notice that the elements of the family $\mathcal{T} = \cup_j F_j$ and the combinations of them will play the role of functionals belonging to a norming set. This explains the use of e_k^* instead of e_k. For simplicity, we also assume that the families $(F_j)_j$ are pairwise disjoint. This happens in the aforementioned example although it is not always true. Under this additional assumption to each $\phi \in \cup_j F_j$ corresponds a unique index by the rule $\text{ind}(\phi) = j$ iff $\phi \in F_j$. Further for a finite block sequence $s = (\phi_1, \dots, \phi_d)$ with each $\phi_i \in \cup_j F_j$, we define $\text{ind}(s) = \{\text{ind}(\phi_1), \dots, \text{ind}(\phi_d)\}$.

The σ-coding: Let Ω_1, Ω_2 be a partition of $\mathbb{N}$ into two infinite disjoint subsets. We denote by $\mathcal{S}$ the family of all block sequences $s = (\phi_1 < \phi_2 < \dots < \phi_d)$ such that $\phi_i \in \cup_j F_j$, $\text{ind}(\phi_1) \in \Omega_1$, $\{\text{ind}(\phi_2) < \dots < \text{ind}(\phi_d)\} \subset \Omega_2$. Clearly $\mathcal{S}$ is countable, hence there exists an injection

$$\sigma : \mathcal{S} \to \Omega_2$$

satisfying $\sigma(s) > \text{ind}(s)$ for every $s \in \mathcal{S}$.

The σ-special sequences: A sequence $s = (\phi_1 < \phi_2 < \dots < \phi_n)$ in $\mathcal{S}$ is said to be a σ-special sequence iff for every $1 \leq i < n$ setting $s_i = (\phi_1 < \dots < \phi_i)$ we have that

$$\phi_{i+1} \in F_{\sigma(s_i)}.$$

The following tree-like interference holds for σ-special sequences.

Let s, t be two σ-special sequences with $s = (\phi_1, \dots, \phi_n)$, $t = (\psi_1, \dots, \psi_m)$. We set

$$i_{s,t} = \max\{i : \text{ind}(\phi_i) = \text{ind}(\psi_i)\}$$

if the later set is non empty. Otherwise we set $i_{s,t} = 0$. Then the following are easily checked.

(a) For every $i < i_{s,t}$ we have that $\phi_i = \psi_i$.
(b) $\{\text{ind}(\phi_i) : i > i_{s,t}\} \cap \{\text{ind}(\psi_j) : j > i_{s,t}\} = \emptyset$.

These two properties immediately yield that the set $\mathcal{T} \cup_j F_j$ endowed with the partial order $\phi \prec_\sigma \psi$ iff there exists a σ-special sequence $(\phi_1, \dots, \phi_n)$ and $1 \le i < j \le n$ with $\phi = \phi_i$ and $\psi = \phi_j$ is a tree.

Now for the given tree structure $(\mathcal{T}, \prec_\sigma)$ we will define norms similar to the classical James tree norm mentioned above.

The space $JT_{\mathcal{F}_2}$: For the first application the family $(F_j)_j$ is the one defined above.

For a σ-special sequence $s = (\phi_1, \dots, \phi_n)$ and an interval E of $\mathbb{N}$ we set $s^* = \sum_{k=1}^n \phi_k$ and let Es^* be the restriction of s^* on E (or the pointwise product $s^*\chi_E$). A σ-special functional x^* is any element Es^* as before.

Also, for a σ-special functional $x^* = Es^*$, $s = (\phi_1, \dots, \phi_n)$, we let $\text{ind}(x^*) = \{\text{ind}(\phi_k) : \text{supp}\,\phi_k \cap E \neq \emptyset\}$. We consider the following set

$$\mathcal{F}_2 = \{\pm e_n^* : n \in \mathbb{N}\} \cup \{\sum_{i=1}^d a_i x_i^* : a_i \in \mathbb{Q}, \sum_{i=1}^d a_i^2 \le 1, (x_i^*)_{i=1}^d \text{ are}$$
$$\sigma\text{-special functionals with } (\text{ind}(x_i^*))_{i=1}^d \text{ pairwise disjoint}, d \in \mathbb{N}\}$$

The space $JT_{\mathcal{F}_2}$ is the completion of $(c_{00}, \|.\|_{\mathcal{F}_2})$ where for $x \in c_{00}$,

$$\|x\|_{\mathcal{F}_2} = \sup\{\phi(x) : \phi \in \mathcal{F}_2\}.$$

Comparing the norming set $\mathcal{F}_2$ with the norming set G_{JT} of JT one observes that σ-special functionals in $\mathcal{F}_2$ play the role of the functionals s^* defined by the segments of the dyadic tree $\mathcal{D}$. As we have mentioned in Proposition 0.2, the space $JT_{\mathcal{F}_2}$, like JT, is ℓ_2 saturated, but for every $M \in [\mathbb{N}]$, the subspace $X_M \overline{\text{span}}\{e_n : n \in M\}$ has non separable dual. The later is a consequence of the fact that the tree structure $(\mathcal{T}, \prec_\sigma)$ is richer than that of the dyadic tree basis in JT. Indeed, it is easy to check that for every $M \in [\mathbb{N}]$ we can construct a block sequence $(\phi_a)_{a\in\mathcal{D}}$ such that

(i) $\phi_a = \frac{1}{m_{j_a}^2} \sum_{k\in F_a} e_k^*$ where $\#F_a = n_{j_a}$ and $F_a \subset M$, while $F_a < F_\beta$ if $a \prec_{lex} \beta$.
(ii) For a branch $b = (a_1 \prec a_2 \prec \cdots \prec a_n \prec \cdots)$ of $\mathcal{D}$ and for every $n \in \mathbb{N}$ we have that $(\phi_{a_1}, \dots, \phi_{a_n})$ is a σ-special sequence.

Defining now $x_a = \frac{m_{j_a}^2}{n_{j_a}} \sum_{k\in F_a} e_k$, the family $(x_a)_{a\in\mathcal{D}}$ provides the James tree structure of X_M.

The Gowers Tree space. The definition of $\mathfrak{X}_{gt}$ uses similar ingredients with the corresponding of $JT_{\mathcal{F}_2}$ although structurally the two spaces are entirely different. The norming set G_{gt} of Gowers space is saturated under the operations $(\mathcal{A}_{n_j}, \frac{1}{m_j})_j$. We recall that a subset G of c_{00} is closed (or saturated) for the operation $(\mathcal{A}_n, \frac{1}{m})$ if for every $\phi_1 < \phi_2 < \cdots < \phi_k$, $k \le n$ with $\phi_i \in G$, $i = 1, \dots, k$, the functional $\phi = \frac{1}{m} \sum_{i=1}^k \phi_i$ belongs to G.

The norming set G_{gt} is the minimal subset of c_{00} satisfying the following conditions:

(i) $\{\pm e_k^* : k \in \mathbb{N}\} \subset G_{gt}$, G_{gt} is symmetric and closed under the operation of restricting elements to the intervals.

(ii) G_{gt} is closed in the $(\mathcal{A}_{n_j}, \frac{1}{m_j})_j$ operations. We also set

$$K_j = \{\phi \in G_{gt} : \phi \text{ is the result of a } (\mathcal{A}_{n_j}, \frac{1}{m_j}) \text{ operation}\}$$

(iii) G_{gt} contains the set

$$\{\sum_{i=1}^{d} a_i x_i^* : \ a_i \in \mathbb{Q}, \ \sum_{i=1}^{d} a_i^2 \leq 1, \ (x_i^*)_{i=1}^d, \ \sigma\text{-special functionals}$$
$$\text{with } (\text{ind}(x_i^*))_{i=1}^d \text{ pairwise disjoint}, \ d \in \mathbb{N}\}$$

(iv) G_{gt} is rationally convex.

We explain briefly condition (iii). For a coding σ, the σ-special sequences $(\phi_1, \ldots, \phi_n)$ are defined as in the case of $\mathcal{F}_2$. Here the set K_j plays the role of the corresponding F_j in $\mathcal{F}_2$. The σ-special functionals x^*, are defined as in the case of $\mathcal{F}_2$.

Let's observe that G_{gt} is almost identical with $\mathcal{F}_2$, although the spaces defined by them are entirely different. The essential difference between $\mathcal{F}_2$ and G_{gt} is that in the case of $\mathcal{F}_2$ each F_j, $j \in \mathbb{N}$ does not norm any subspace of $JT_{\mathcal{F}_2}$, while in $\mathfrak{X}_{gt}$ each K_j defines an equivalent norm on $\mathfrak{X}_{gt}$. The later means that in every subspace Y of $\mathfrak{X}_{gt}$, the families K_j, $j \in \mathbb{N}$ as well as $\{x^* : x^* \text{ is a } \sigma\text{-special functional}\}$ and $\{\sum_{i=1}^{d} \lambda_i x_i^* : \sum_{i=1}^{n} \lambda_i^2 \leq 1, \ (\text{ind}(x_i^*))_{i=1}^d \text{ are pairwise disjoint}\}$ define equivalent norms making it difficult to distinguish the action of them on the elements of Y. Thus, while the spaces of the form $JT_{\mathcal{F}_2}$ can be studied in terms of the classical theory, the space $\mathfrak{X}_{gt}$ requires advanced tools, like Gowers probabilistic argument, which do not permit a complete understanding of its structure.

0.2. **Saturated extensions.** The method of HI extensions appeared in the Memoirs monograph [AT1] and was used to derive the following two results:

- Every separable Banach space Z not containing ℓ_1 is a quotient of a separable HI space X, with the additional property that Q^*Z^* is a complemented subspace of X^*. (Here Q denotes the quotient map from X to Z.)
- There exists a nonseparable HI Banach space.

Roughly speaking, the method of HI extensions provides a tool to connect a given norm, usually defined through a norming set G with a HI norm. The resulting new norm will preserve some of the ingredients of the initial norm and will also be HI. To some extent, HI extensions, have similar goals with HI interpolations ([AF]) and some of the results could be obtained with both methods. However it seems that the method of extensions is very efficient when we want to construct dual pairs X, X^* with divergent structure. This actually requires the combination of extensions with the method of attractors, which appeared in [AT2] where a reflexive HI space X is constructed with X^* unconditionally saturated.

In the sequel we shall provide a general definition of saturated extensions which include several forms of extensions which appeared elsewhere (c.f. [AT1, AT2, ArTo])

Let $\mathcal{M}$ be a compact family of finite subsets of $\mathbb{N}$. For the purposes of the present paper, $\mathcal{M}$ will be either some $\mathcal{A}_n = \{F \subset \mathbb{N} : \#F \leq n\}$, or some $\mathcal{S}_n$, the n^{th} Schreier family. For $0 < \theta < 1$, the $(\mathcal{M}, \theta)$-operation on c_{00} is a map which assigns to each $\mathcal{M}$-admissible block sequence $(\phi_1 < \phi_2 < \cdots < \phi_n)$, the functional $\theta \sum_{i=1}^n \phi_i$. (We recall that $\phi_1, \phi_2, \ldots, \phi_n$ is $\mathcal{M}$-admissible if $\{\min \operatorname{supp} \phi_i : i = 1, \ldots, n\}$ belongs to $\mathcal{M}$.) A subset G of c_{00} is said to be closed in the $(\mathcal{M}, \theta)$-operation, if for every $\mathcal{M}$-admissible block sequence $\phi_1, \ldots, \phi_n$, with each $\phi_i \in G$, the functional $\theta \sum_{i=1}^n \phi_i$ belongs to G. When we refer to saturated norms we shall mean that there exists a norming set G which is closed under certain $(\mathcal{M}_j, \theta_j)_j$ operations.

Let G be a subset of c_{00}. The set G is said to be a ground set if it is symmetric, $\{e_n^* : n \in \mathbb{N}\}$ is contained in G, $\|\phi\|_\infty \leq 1$, $\phi(n) \in \mathbb{Q}$ for all $\phi \in G$ and G is closed under the restriction of its elements to intervals of $\mathbb{N}$. A ground norm, $\|\cdot\|_G$ is any norm induced on c_{00} by a ground set G. It turns out that for every space $(X, \|\cdot\|_X)$ with a normalized Schauder basis $(x_n)_n$ there exists a ground set G_X such that the natural map $e_n \mapsto x_n$ defines an isomorphism between $(X, \|\cdot\|_X)$ and $\overline{(c_{00}, \|\cdot\|_{G_X})}$.

Saturated extensions of a ground set G. Let G be a ground set, $(m_j)_j$ an appropriate sequence of natural numbers and $(\mathcal{M}_j)_j$ a sequence of compact families such that $(\mathcal{M}_j)_j$ is either $(\mathcal{A}_{n_j})_j$ or $(S_{n_j})_j$.

Denote by E_G the minimal subset of c_{00} such that

(i) The ground set G is a subset of E_G.
(ii) The set E_G is closed in the $(\mathcal{M}_j, \frac{1}{m_j})$ operation.
(iii) The set E_G is rationally convex.

Definition 0.3. A subset D_G of E_G is said to be a saturated extension of the ground set G if the following hold:

(i) The set D_G is a subset of E_G, the ground set G is contained in D_G and D_G is closed under restrictions of its elements to intervals.
(ii) The set D_G is closed under even operations $(\mathcal{M}_{2j}, \frac{1}{m_{2j}})_j$.
(iii) The set D_G is rationally convex.
(iv) Every $\phi \in D_G$ admits a tree analysis $(f_t)_{t \in T}$ with each $f_t \in D_G$.

Denoting by $\|\cdot\|_{D_G}$ the norm on c_{00} induced by D_G and letting X_{D_G} be the space $\overline{(c_{00}, \|\cdot\|_{D_G})}$, we call X_{D_G} a *saturated extension* of the space $X_G = \overline{(c_{00}, \|\cdot\|_G)}$.

Let's point out that the basis $(e_n)_n$ of c_{00} is a bimonotone boundedly complete Schauder basis of X_{D_G} and that the identity $I : X_{D_G} \to X_G$ is a norm one operator. Observe also that we make no assumption concerning the odd operations. As we will see later making several assumptions for the odd operations, we will derive saturated extensions with different properties.

A last comment on the definition of D_G, is related to the condition (iv). The tree analysis $(f_t)_{t\in T}$ of a functional f in E_G describes an inductive procedure for obtaining f starting from elements of the ground set G and either applying operations $(\mathcal{M}_j, \frac{1}{m_j})$ or taking rational convex combinations. This tree structure is completely irrelevant to the tree structures discussed in the previous subsection. Its role is to help estimate upper bounds of the norm of vectors in X_{D_G}.

Properties and variants of Saturated extensions.

As we have mentioned, for $x \in c_{00}$, $\|x\|_G \leq \|x\|_{D_G}$. This is an immediate consequence of the fact that $G \subset D_G$. On the other hand, there are cases of ground sets G such that D_G does not add more information beyond G itself. Such a case is when G defines a norm $\|\cdot\|_G$ equivalent to the ℓ_1 norm. The measure of the fact that $\|\cdot\|_{D_G}$ is strictly greater than $\|\cdot\|_G$ on a subspace Y of X_{D_G} is that the identity operator $I : X_{D_G} \to X_G$ restricted to Y is a strictly singular one. If $I : X_{D_G} \to X_G$ is strictly singular, then we refer to strictly singular extensions. The first result we want to mention is that strictly singular extensions are reflexive ones. More precisely the following holds:

Proposition 0.4. Let Y be a closed subspace of X_{D_G} such that $I|_Y : Y \to X_G$ is strictly singular. Then Y is reflexively saturated. In particular X_{D_G} is reflexively saturated whenever it is a strictly singular extension.

Next we proceed to specify the odd operations and to derive additional information on the structure of X_{D_G} whenever X_{D_G} is a strictly singular extension.

(a) Unconditionally saturated extensions.

This is the case where $D_G = E_G = D_G^u$. In this case the following holds:

Proposition 0.5. Let Y be a closed subspace of $X_{D_G^u}$ such that $I|_Y : Y \to X_G$ is strictly singular. Then Y is unconditionally (and reflexively) saturated.

(b) Hereditarily Indecomposable extensions.

HI extensions, are the most important ones. In this case the norming set D_G^{hi} is defined as follows. D_G^{hi} is the minimal subset of c_{00} satisfying the following conditions

(i) $\{e_n^* : n \in \mathbb{N}\} \subset D_G^{hi}$, D_G^{hi} is symmetric and closed under restriction of its elements to intervals.
(ii) D_G^{hi} is closed under $(\mathcal{M}_{2j}, \frac{1}{m_{2j}})_j$ operations.
(iii) For each j, D_G^{hi} is closed under $(\mathcal{M}_{2j-1}, \frac{1}{m_{2j-1}})$ operation on $2j-1$ special sequences.
(iv) D_G^{hi} is rationally convex.

The $2j-1$ special sequences are defined through a coding σ and satisfy the following conditions.

(a) $(f_1, \ldots, f_d)$ is $\mathcal{M}_{2j-1}$ admissible

(b) For $i \leq i \leq d$ there exists some $j \in \mathbb{N}$ such that $f_i \in K_{2j} = \{\frac{1}{m_{2j}} \sum_{l=1}^{k} \phi_l \; : \; \phi_1 < \cdots < \phi_k$ is $\mathcal{M}_{2j}$ admissible, $\phi_l \in D_G^{hi}\}$ and if $i > 1$ then $2j = \sigma(f_1, \ldots, f_{i-1})$.

Notice that in the definition of D_G^{hi} we do not refer to the tree analysis. The reason is that the existence of a tree analysis follows from the minimality of D_G^{hi} and the conditions involved in its definition.

The analogue of the previous results also holds for HI extensions.

Proposition 0.6. Let Y be a closed subspace of $X_{D_G^{hi}}$ such that $I|_Y : Y \to X_G$ is strictly singular. Then Y is a HI space. In particular strictly singular and HI extensions yield HI spaces.

The above three propositions indicate that if we wish to have additional structure on $X_{D_G}, X_{D_G^u}, X_{D_G^{hi}}$ we need to consider strictly singular extensions. As is shown in [AT1], this is always possible. Indeed, for every ground set G such that the corresponding space X_G does not contain ℓ_1 there exists a family $(\mathcal{M}_j, \frac{1}{m_j})_j$ such that the saturated extension of G by this family is a strictly singular one. Thus the following is proven ([AT1]).

Theorem 0.7. Let X be a Banach space with a normalized Schauder basis $(x_n)_n$ such that X contains no isomorphic copy of ℓ_1. Then there exists a HI space Z with a normalized basis $(z_n)_n$ such that the map $z_n \mapsto x_n$ has a linear extension to a bounded operator $T : Z \to X$.

This theorem in conjunction with the following one yields that every separable Banach space X not containing ℓ_1 is the quotient of a HI space.

Theorem 0.8 ([AT1])**.** Let X be a separable Banach space not containing ℓ_1. Then there exists a space Y not containing ℓ_1, with a normalized Schauder basis $(y_n)_n$ and a bounded linear operator $T : Y \to X$ such that $(Ty_n)_n$ is a dense subset of the unit sphere of X.

The predual $(X_{D_G^{hi}})_*$. As we have mentioned before the basis $(e_n)_{n\in\mathbb{N}}$ of $X_{D_G^{hi}}$ is boundedly complete, hence the space $(X_{D_G^{hi}})_*$, which is the subspace of $X^*_{D_G^{hi}}$ norm generated by the biorthogonal functionals $(e_n^*)_{n\in\mathbb{N}}$, is a predual of $X_{D_G^{hi}}$. In many cases it is shown that $(X_{D_G^{hi}})_*$ is also a HI space. This requires some additional information concerning the weakly null block sequences in X_G, which is stronger than the assumption that the identity map $I : X_{D_G^{hi}} \to X_G$ is strictly singular. For example in [AT1], for extensions using the operations $(\mathcal{S}_{n_j}, \frac{1}{m_j})_j$, had been assumed that the ground set G is $\mathcal{S}_2$ bounded. In the present paper for the operations $(\mathcal{A}_{n_j}, \frac{1}{m_j})_j$ we introduce the concept of strongly strictly singular extension which yields that $(X_{D_G^{hi}})_*$ is HI. It is also worth pointing out that $(X_{D_G^{hi}})_*$ is not necessarily reflexively saturated as happens for the strictly singular extensions X_{D_G} $X_{D_G^{hi}}$. This actually will be a key point in our approach for constructing HI spaces with no reflexive subspace.