ITERATION OF INNER FUNCTIONS AND BOUNDARIES OF COMPONENTS OF THE FATOU SET

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Abstract. Let D be an unbounded invariant component of the Fatou set of a transcendental entire function f. Let $\phi : \mathbb{D} \to D$ be a Riemann map. Then the set $\Theta := \{\theta \in \partial\mathbb{D} : \lim_{r\to 1} \phi(r\theta) = \infty\}$ is closely related to the Julia set of the corresponding inner function $g := \phi^{-1} \circ f \circ \phi$. In the first part of the paper we further develop the theory of Julia sets of inner functions and the dynamical behaviour on their Fatou sets. In the second part we apply these results to iteration of entire functions by using the above relation and obtain some new results about the boundaries of components of the Fatou set of an entire function.

1. Introduction

Dynamics of inner functions have turned out to be a very useful tool to study the boundary structure of unbounded invariant components of the Fatou set of a transcendental entire function. The key method in this area has been developed by I.N. Baker and P. Domínguez [3]. We recall some of their techniques and results in Subsection 1.1 and then state the aims and results of this paper in Subsection 1.2

1.1. The method of Baker-Domínguez.

Let f be a transcendental entire function, with Fatou set $\mathcal{F}(f)$ and Julia set $\mathcal{J}(f)$; see [7] for background information on these concepts. Suppose that D is an unbounded invariant component of $\mathcal{F}(f)$. Then D is simply connected [2, Theorem 1], which implies that there exists a biholomorphic (Riemann) map $\phi : \mathbb{D} \to D$. Then

$$g := \phi^{-1} \circ f \circ \phi$$

is an inner function, i.e. a holomorphic self-map of the unit disk such that

$$\lim_{r\to 1} |g(r \exp(2\pi i\alpha))| = 1,$$

for almost every $\alpha \in [0, 1]$. According to Fatou's theorem [18, p. 139] this implies that $\lim_{r\to 1} g(r \exp(2\pi i\alpha))$ exists and is contained in $\partial\mathbb{D}$, for almost every $\alpha \in [0, 1]$.

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If $f|_D$ is a proper self-map of D, then g is a (finite) Blaschke product, i.e. there exist $m, n \in \mathbb{N}_0$, $\lambda \in \partial\mathbb{D}$, and $a_1, \ldots, a_n \in \mathbb{D} \setminus \{0\}$ such that

$$g(z) = \lambda z^m \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z},$$

for each $z \in \mathbb{D}$.

Since D is unbounded $f|_D$ need not be a proper self-map of D. In this case g has at least one singularity on the boundary of the unit disk.

Definition 1.1. *Let g be an inner function of $\mathbb{D}$. A point $\zeta \in \partial\mathbb{D}$ is called a singularity of g if g cannot be continued analytically to a neighbourhood of ζ. Denote the set of all singularities of g by* $\operatorname{sing}(g)$.

Throughout this paper we assume that an inner function is always continued to $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ by the reflection principle, where $\hat{\mathbb{C}}$ denotes the complex sphere, and to $\partial\mathbb{D} \setminus \operatorname{sing}(g)$ by analytic continuation.

It follows from the theory of inner functions that the composition of two inner functions is again an inner function; see [3, Lemma 4]. In particular, the n-th iterate g^n of an inner function g is an inner function. Now, the Julia set of an inner function can be defined in the following way.

Definition 1.2. *Let g be an inner function of the unit disk $\mathbb{D}$. The Fatou set $\mathcal{F}(g)$ of g is the set of all points $z \in \hat{\mathbb{C}}$ for which there is an open neighbourhood $U \subset \hat{\mathbb{C}}$ of z such that $U \cap \operatorname{sing}(g^n) = \emptyset$, for each $n \in \mathbb{N}$, and $\{g^n|_U : n \in \mathbb{N}\}$ is normal. The Julia set $\mathcal{J}(g)$ of g is the complement of $\mathcal{F}(g)$ in $\hat{\mathbb{C}}$.*

Remark 1.3. *It follows from Montel's theorem that $\mathcal{J}(g) \subset \partial\mathbb{D}$. Moreover, for the case of a finite Blaschke product this definition coincides with the usual definition of the Julia set of a rational function.*

Baker and Domínguez initiated the study of Julia sets of inner functions by proving the following result [3, Lemma 8].

Theorem 1.4 (Baker-Domínguez)**.** *Let g be an inner function of the unit disk $\mathbb{D}$. Then the following properties hold.*

(1) *$g(\mathcal{F}(g)) \subset \mathcal{F}(g)$.*
(2) *If g is non-Möbius, then $\mathcal{J}(g)$ is a perfect set.*
(3) *If g is non-rational, then $\mathcal{J}(g) = \overline{\bigcup_{n\in\mathbb{N}} \operatorname{sing}(g^n)}$.*

The main tool in the proof of Theorem 1.4 is the the following lemma on inner functions [3, Lemma 5], which we use later. For the definition of a Stolz angle we refer the reader to [18, p. 6].

Lemma 1.5 (Baker-Domínguez)**.** *Let g be an inner function of the unit disk. Suppose that $\zeta \in \operatorname{sing}(g)$. Then, for each $\theta \in \partial\mathbb{D}$ and each neighbourhood U of ζ, there exist $\eta \in U \setminus \{\zeta\}$ and a path $\gamma : [0, 1) \to \mathbb{D}$ such that $\lim_{t\to 1} \gamma(t) = \eta$ and $g(\gamma(t)) \to \theta$ in a Stolz angle as $t \to 1$.*

Following Baker and Domínguez [3], we relate the boundary behaviour of the Riemann map ϕ to the dynamical behaviour of the corresponding inner function g. This process involves the sets

$$\Xi := \{\zeta \in \partial\mathbb{D} : \infty \in C(\phi;\zeta)\},$$

and

$$\Theta := \{\theta \in \partial\mathbb{D} : \lim_{r\to 1} \phi(r\theta) = \infty\}.$$

Here $C(\phi;\zeta)$ denotes the cluster set of ϕ at ζ, i.e. the set of all values $w \in \hat{\mathbb{C}}$ for which there is sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{D}$ such that $z_n \to \zeta$ and $\phi(z_n) \to w$ as $n \to \infty$. Since D is unbounded the set Ξ is always non-empty. In general, it is not known whether the set Θ is always non-empty. However, for the case when D is a Baker domain, i.e. $f^n|_D \to \infty$ locally uniformly, it is easy to see that $\Theta \neq \emptyset$. Throughout this paper the sets Ξ and Θ will relate to a Riemann mapping $\phi : \mathbb{D} \to D$ of the invariant Fatou component D of the function f under consideration.

There is a close connection between Ξ and $\mathcal{J}(g)$.

Lemma 1.6. *If $f|_D$ is not an automorphism of D, then $\mathcal{J}(g) \subset \Xi$.*

Proof. First, it is easy to see that Ξ is closed.

Case 1. Suppose that g is rational. Then g is a finite Blaschke product and it is easy to see that Ξ is backward invariant under g. Since g is locally injective on $\partial\mathbb{D}$ (see Remark 2.19) we conclude that Ξ is an infinite set. Hence Ξ is a closed, backward invariant set which contains at least three points. This implies that Ξ is a superset of $\mathcal{J}(g)$.

Case 2. Suppose that g is not rational. By Theorem 1.4 we need only show that the singularities of the iterates of g are contained in Ξ. Let $n \in \mathbb{N}$ and ζ be a singularity of g^n. Then $C(g^n;\zeta) = \overline{\mathbb{D}}$ (see for instance [13, Theorem 5.4]), which implies that

$$\infty \in \overline{\mathbb{D}} \subset C(\phi \circ g^n;\zeta) = C(f^n \circ \phi;\zeta).$$

Thus we conclude that $\infty \in C(\phi;\zeta)$. □

Using Lemma 1.5, Baker and Domínguez obtained a similar result for the set Θ; see [3, Lemma 13].

Lemma 1.7 (Baker-Domínguez)**.** *Suppose that $f|_D$ is not an automorphism of D and $\Theta \neq \emptyset$. Then $\mathcal{J}(g) \subset \overline{\Theta}$.*

Hence the Julia set of the corresponding inner function g is a lower bound for the size of the set Ξ and, provided that $\Theta \neq \emptyset$, for the size of the set $\overline{\Theta}$. This provides a strategy to show that the sets Ξ and $\overline{\Theta}$ are equal to the unit circle by showing that the Julia sets of the corresponding inner functions are the unit circle. Baker and Domínguez gave two cases when $\mathcal{J}(g) = \partial\mathbb{D}$; see [3, Lemma 9 and Lemma 10].

Theorem 1.8 (Baker-Domínguez). *Let g be a non-Möbius inner function of $\mathbb{D}$ with a fixed point $p \in \mathbb{D}$. Then $\mathcal{J}(g) = \partial\mathbb{D}$.*

Theorem 1.9 (Baker-Domínguez). *Let g be an inner function of $\mathbb{D}$. Suppose that there exists $p \in \partial\mathbb{D}$ such that, for each $z \in \mathbb{D}$, $g^n(z) \to p$ in an arbitrarily small Stolz angle as $n \to \infty$. Then g is non-Möbius and $\mathcal{J}(g) = \partial\mathbb{D}$.*

The assumptions of Theorem 1.8 are satisfied for the corresponding inner function g when D is an attracting domain of f. Theorem 1.9 can be applied to the corresponding inner function g when D is a parabolic domain of f. Taken together, these lead to the following result; see [4, Theorem 1] and [3, Theorem 1.1].

Theorem 1.10 (Baker-Weinreich, Baker-Domínguez). *Suppose that D is an attracting domain, a parabolic domain, or a Siegel disk of f. Then*

$$\Xi = \partial\mathbb{D} \quad \textit{and} \quad \Theta \neq \emptyset \Rightarrow \overline{\Theta} = \partial\mathbb{D}.$$

This result does not carry over in general to Baker domains, since there are examples of Baker domains (due to Baker-Weinreich and Bergweiler, see Examples 3.4 and 3.5 in Subsection 3.1) whose boundaries are Jordan curves. On the other hand, Baker and Weinreich proved [4, Theorem 4] that if D is a Baker domain whose boundary is a Jordan curve, then $f|_D$ has to be univalent. Thus it is natural to ask whether Theorem 1.10 holds for those Baker domains for which $f|_D$ is not univalent. In this case, Baker and Domínguez proved that Θ contains a perfect set and hence is infinite; see [3, Theorem 1.2].

1.2. **Aims and results of this paper.** The aim of this paper is twofold. In Section 2 we further develop the theory of Julia sets of inner functions, independently of the application to iteration of entire functions. Then, in Section 3, we apply these results to iteration of entire functions. Here, we use the method of Baker-Domínguez described in Subsection 1.1 and further extend it.

In Section 2, the main results are in Subsections 2.3 and 2.4. In Subsection 2.3 we prove the following theorem, which is a generalization of Theorems 1.8 and 1.9. For a hyperbolic domain G in the complex sphere $\hat{\mathbb{C}}$, λ_G always denotes the hyperbolic metric on G.

Theorem 2.24 *Let g be an inner function such that $\lambda_{\mathbb{D}}(g^n(z), g^{n+1}(z)) \to 0$ as $n \to \infty$ for some $z \in \mathbb{D}$. Then $\mathcal{J}(g) = \partial\mathbb{D}$.*

Theorem 2.24 will be an easy consequence of two more general theorems which give necessary and sufficient conditions for an inner function to be eventually conjugated to a certain Möbius transformation on the Fatou set and which also classify the possible eventual conjugacies which can arise. See Subsection 2.1 for the meaning of 'eventual conjugacy'.

The different types of components of $\mathcal{F}(g) \cap \partial\mathbb{D}$ will be introduced and classified in Subsection 2.2.

In Subsection 2.4 we prove that the Julia set of an inner function coincides with the closure of the set of repelling periodic points (with a suitable definition of repelling periodic point, see Definition 2.30). Analogous results are known for rational and entire functions and it is an interesting fact that inner functions have this property, too.

Theorem 2.34 *Let g be a non-Möbius inner function. Then $\mathcal{J}(g)$ is the closure of the set of the repelling periodic points of g.*

At the end of Section 2, we give some examples of inner functions; in particular, we show that all the possible types of eventual conjugacy can occur.

In Section 3, we apply the results of Section 2 to the iteration of entire functions. For instance, using Lemma 1.7 and our Theorem 2.24 we can prove the following generalization of Theorem 1.10.

Theorem 3.2 *Let f be a transcendental entire function. Suppose that D is an unbounded invariant component of the Fatou set of f such that*

$$\lambda_D(f^n(z), f^{n+1}(z)) \to 0 \quad \text{as } n \to \infty,$$

for some $z \in D$. If $\Theta \neq \emptyset$, then $\overline{\Theta} = \partial\mathbb{D}$.

We can use Theorem 3.2 to extend Theorem 1.10 to a certain class of Baker domains. Here we use the symbol $\sim$ to indicate an eventual conjugacy.

Theorem 3.1 *Let f be a transcendental entire function. Suppose that D is a Baker domain of f such that $f|_D \sim \mathrm{id}_{\mathbb{C}} + 1$. Then $\overline{\Theta} = \partial\mathbb{D}$.*

Moreover, we see in Subsection 3.1 that, for a whole class of examples, the set Θ is dense in $\partial\mathbb{D}$ whenever f is not univalent on the Baker domain D; see Lemma 3.3.

In Subsection 3.2 we improve Lemma 1.7, at least for the case when D is a completely invariant component of the Fatou set.

Theorem 3.8 *Let f be a transcendental entire function. Suppose that D is a completely invariant component of the Fatou set of f. Let $\phi : \mathbb{D} \to D$ be a Riemann map and let $g := \phi^{-1} \circ f \circ \phi$ be the corresponding inner function. If $\Theta \neq \emptyset$, then $\mathcal{J}(g)$ is equal to the set of accumulation points of Θ.*

As a consequence of this result, we shall be able to prove the following results about boundaries of components of the Fatou set; see Subsection 3.3.

Theorem 3.11 *Let f be a transcendental entire function. Suppose that D is a completely invariant component of the Fatou set of f. Let $G \subset \mathbb{C}$ be a bounded Jordan domain such that $G \cap \mathcal{J}(f) \neq \emptyset$. Then $\partial G \cap D$ has infinitely many components.*

Theorem 3.12 *Let f be a transcendental entire function. Suppose that there exists an unbounded component of the Fatou set of f. Let $G \subset \mathbb{C}$ be a bounded Jordan domain such that $G \cap \mathcal{J}(f) \neq \emptyset$. Then $\partial G \cap \mathcal{F}(f)$ has infinitely many components.*

Theorem 3.14 *Let f be an entire function. Let D be a component of the Fatou set of f. Suppose that at least one of the following conditions is satisfied:*

(1) *f is transcendental and there exists an unbounded component of the Fatou set of f, or*
(2) *$\bigcup_{n\in\mathbb{N}} f^n(D)$ is bounded.*

Let $\phi : \mathbb{D} \to D$ be a Riemann map. Let $\mathrm{Acc}(D)$ be the set of finite accessible boundary points of D, and let Z be the set of all $\zeta \in \partial\mathbb{D}$ such that $\phi(\zeta) := \lim_{r\to 1} \phi(r\zeta)$ exists and is finite. Then the map

$$Z \to \mathrm{Acc}(D),\ \zeta \mapsto \phi(\zeta)$$

is a bijection.

Corollary 3.15 *Let f be an entire function. Suppose that D is a Siegel disk for f. Then*

(1) *There is no periodic point of f in ∂D which is an accessible boundary point of D.*
(2) *f is univalent on the set of finite accessible boundary points of D.*

Theorem 3.16 *Let f be an entire function. Let $\mathcal{E}$ be a finite set of components of the Fatou set of f. Suppose that at least one of the following conditions is satisfied:*

(1) *f is transcendental and there exists an unbounded component of the Fatou set of f, or*
(2) *$\bigcup_{n\in\mathbb{N}} f^n(D)$ is bounded, for each $D \in \mathcal{E}$.*

Then there are at most $\mathrm{card}(\mathcal{E}) - 1$ points in $\mathbb{C}$ which are common accessible boundary points of at least two components in $\mathcal{E}$.

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2. Iteration of inner functions

2.1. Holomorphic self-maps of hyperbolic domains. In this subsection we recall some facts about the dynamical behaviour of a holomorphic self-map of the unit disk or, more generally, of a hyperbolic domain in the complex sphere. We start with an old theorem of A. Denjoy [15] and J. Wolff [22]; see also [12, p. 79] or [21, p. 43]. For the notion of the angular limit we refer the reader to [18, p. 6].

Theorem 2.1 (Denjoy-Wolff)**.** *Let h be a non-Möbius holomorphic self-map of $\mathbb{D}$. Then there is a point $p \in \overline{\mathbb{D}}$ such that $h^n \to p$ locally uniformly on $\mathbb{D}$. Moreover, if $p \in \partial\mathbb{D}$, then h has the angular limit p at p.*

The point p referred to in this theorem is often called the Denjoy-Wolff point of h. It may appear that the dynamical behaviour of a holomorphic self-map depends only on the question whether its Denjoy-Wolff point p is inside the disk or on its boundary. But, in fact, the case $p \in \partial\mathbb{D}$ can be further subdivided. Here we make use of a classification due to C. Cowen [14] who has shown that a holomorphic self-map of the unit disk is eventually conjugated to a certain Möbius transformation. Roughly speaking, eventually conjugated means that the function is semi-conjugated to a Möbius transformation and, starting at an arbitrary point and iterating, one eventually lands in a simply connected region where the function is even conjugated to this Möbius transformation. More precisely, we have the following definition.

Definition 2.2. *Let h be a holomorphic self-map of a hyperbolic domain $G \subset \hat{\mathbb{C}}$. Let T be a biholomorphic self-map of a simply connected domain $\Omega \subset \mathbb{C}$. Then we say that*

$$h \sim T \quad (h \textit{ is eventually conjugated to } T)$$

if there exist a holomorphic function $\Phi : G \to \Omega$ and a simply connected domain $V \subset G$ such that the following conditions are satisfied:

(1) $\Phi \circ h = T \circ \Phi$.
(2) Φ *is univalent on* V.
(3) V *is a fundamental set for* h *on* G, *i.e.*

$$h(V) \subset V \quad \textit{and} \quad \forall\, z \in G\, \exists\, n \in \mathbb{N} : h^n(z) \in V.$$

(4) $\Phi(V)$ *is a fundamental set for* T *on* Ω.

In this case, (Ω, T, Φ, V) is called an eventual conjugacy of h on G.

Eventual conjugacies are unique in the following sense; see [14, p. 79-80].

Lemma 2.3 (Cowen)**.** *Let h be a holomorphic self-map of a hyperbolic domain $G \subset \hat{\mathbb{C}}$. Let $(\Omega_1, T_1, \Phi_1, V_1)$ be an eventual conjugacy of h on G.*

(1) *Let $\tau : \Omega_1 \to \mathbb{C}$ be an injective holomorphic function. Then $(\tau(\Omega_1), \tau \circ T_1 \circ \tau^{-1}, \tau \circ \Phi_1, V_1)$ is an eventual conjugacy of h on G.*
(2) *Let $(\Omega_2, T_2, \Phi_2, V_2)$ be another eventual conjugacy of h on G. Then there exists a component W of $V_1 \cap V_2$ such that W is a fundamental set for h on G and, for each $j \in \{1, 2\}$, $\Phi_j(W)$ is a fundamental set for T_j on Ω_j. Moreover, there exists a biholomorphic map $\tau : \Omega_1 \to \Omega_2$ such that $\Phi_2 = \tau \circ \Phi_1$ and $T_2 = \tau \circ T_1 \circ \tau^{-1}$.*

Cowen's result can be stated as follows. Here, $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

Theorem 2.4 (Cowen). *Let h be a holomorphic self-map of the unit disk $\mathbb{D}$ without a fixed point. Then exactly one of the following statements holds.*

(1) $h \sim \mathrm{id}_{\mathbb{C}} + 1$.
(2) *There exists exactly one* $\sigma \in \{-1, 1\}$ *such that* $h \sim \mathrm{id}_{\mathbb{H}} + \sigma$.
(3) *There exists exactly one* $\lambda > 1$ *such that* $h \sim \lambda \mathrm{id}_{\mathbb{H}}$.

For the proof of Theorem 2.4, see [14, Theorem 3.2] and the following remark.

Remark 2.5. *By applying Lemma 2.3 it is easy to see that, for each $\sigma \in \{-1, 1\}$,*

$$h \sim \mathrm{id}_{\mathbb{H}} + \sigma \iff h \sim \mathrm{id}_{\sigma\mathbb{H}} + 1 \iff h \sim \mathrm{id}_{-i\mathbb{H}} - i\sigma$$

and, for each $\lambda > 1$,

$$h \sim \lambda \mathrm{id}_{\mathbb{H}} \iff h \sim \lambda \mathrm{id}_{-\mathbb{H}} \iff h \sim \lambda \mathrm{id}_{-i\mathbb{H}}.$$

H. König [17] has given geometrical conditions to determine which Möbius transformation the function h is eventually conjugated to. Expressing these conditions in terms of the hyperbolic metric we obtain the following lemma. Recall that we denote the hyperbolic metric on a hyperbolic domain $G \subset \hat{\mathbb{C}}$ by λ_G.

Lemma 2.6. *Let h be a holomorphic self-map of a hyperbolic domain $G \subset \hat{\mathbb{C}}$ without a fixed point. For each $z \in G$, let*

$$\rho(z) := \inf_{n \in \mathbb{N}} \lambda_G(h^n(z), h^{n+1}(z)) = \lim_{n \to \infty} \lambda_G(h^n(z), h^{n+1}(z)).$$

Then we have that

(1) $h \sim \mathrm{id}_{\mathbb{C}} + 1 \Rightarrow \rho = 0$,
(2) $h \sim \mathrm{id}_{\pm\mathbb{H}} + 1 \Rightarrow \rho > 0$ *and* $\inf \rho(G) = 0$,
(3) $\exists\, \lambda > 1 : h \sim \lambda \mathrm{id}_{\mathbb{H}} \Rightarrow \inf \rho(G) > 0$.

Remark 2.7. *If additionally G is simply connected, then Cowen's result implies that all these implications are equivalences. In general, a holomorphic self-map of a hyperbolic domain need not be semi-conjugated to a Möbius transformation at all.*

To prove Lemma 2.6 we make use of the following lemma.

Lemma 2.8. *Let $\Omega \in \{\mathbb{C}, \mathbb{H}, -\mathbb{H}\}$ and let $T := \mathrm{id}_\Omega + 1$. Let $W \subset \Omega$ be a fundamental set for T on Ω. Let $r > 1$ and let $w \in \Omega$ be the center of an open disk Q with radius r such that $\overline{Q} \subset \Omega$. Then*

$$\lim_{n \to \infty} \lambda_W(w + n, w + n + 1) \le \frac{1}{2} \log\left(\frac{r+1}{r-1}\right).$$

Proof. Since W is a fundamental set for $\mathrm{id}_\Omega + 1$ on Ω there exists $n \in \mathbb{N}$ such that $\overline{Q} + n \subset W$. By the Schwarz-Pick lemma we have that

$$\lambda_W(w+n, w+n+1) \leq \lambda_{Q+n}(w+n, w+n+1) = \frac{1}{2}\log\left(1 + \frac{2}{r-1}\right). \quad \square$$

Proof of Lemma 2.6. Let (Ω, T, Φ, V) be an eventual conjugacy of h on G, where $\Omega \in \{\mathbb{C}, \mathbb{H}, -\mathbb{H}\}$. Let $W := \Phi(V)$. For each $z \in G$ and $n \in \mathbb{N}$ such that $h^n(z) \in V$, the Schwarz-Pick lemma implies that

$$(*) \qquad \begin{aligned} \lambda_\Omega(\Phi(z), T(\Phi(z))) &= \lambda_\Omega(T^n(\Phi(z), T^{n+1}(\Phi(z))) \\ &= \lambda_\Omega(\Phi(h^n(z)), \Phi(h^{n+1}(z))) \\ &\leq \lambda_G(h^n(z), h^{n+1}(z)) \\ &\leq \lambda_V(h^n(z), h^{n+1}(z)) \\ &= \lambda_W(\Phi(h^n(z)), \Phi(h^{n+1}(z))) \\ &= \lambda_W(T^n(\Phi(z)), T^{n+1}(\Phi(z))), \end{aligned}$$

where the first inequality makes sense only for the case when $\Omega \in \{\mathbb{H}, -\mathbb{H}\}$.

To prove (3) we observe that, for each $\lambda > 1$ and $w \in \mathbb{H}$,

$$\lambda_{\mathbb{H}}(\lambda w, w) = \log \frac{1 + (\lambda w - w)/(\lambda w - \overline{w})}{1 - (\lambda w - w)/(\lambda w - \overline{w})} \geq \log \lambda,$$

which together with $(*)$ implies that $\inf \rho(G) > 0$ if $h \sim \lambda \mathrm{id}_{\mathbb{H}}$.

To prove (2) suppose that there exists $\sigma \in \{-1, 1\}$ such that $T = \mathrm{id}_{\sigma\mathbb{H}} + 1$. Then, for each $w \in \sigma\mathbb{H}$, we have that

$$\lambda_{\sigma\mathbb{H}}(w+1, w) = \log \frac{1 + |\sigma/(\sigma + 2i\mathrm{Im}(w))|}{1 - |\sigma/(\sigma + 2i\mathrm{Im}(w))|} \geq \log\left(1 + \frac{1}{|\mathrm{Im}(w)|}\right),$$

which together with $(*)$ implies that $\rho > 0$. Because of $(*)$ it remains to show that

$$\inf_{w \in W} \inf_{n \in \mathbb{N}} \lambda_W(w+n, w+n+1) = 0.$$

This is an easy consequence of Lemma 2.8 because $\sigma\mathbb{H}$ contains an open disk with an arbitrarily large radius.

To prove (1) suppose that $T = \mathrm{id}_{\mathbb{C}} + 1$. Because of $(*)$ it remains to show that, for each $w \in \mathbb{C}$,

$$\inf_{n \in \mathbb{N}} \lambda_W(w+n, w+n+1) = 0.$$

This is an easy consequence of Lemma 2.8 because each $w \in \mathbb{C}$ is the center of an open disk in $\mathbb{C}$ with arbitrarily large radius. $\square$

The case when the Denjoy-Wolff point of a holomorphic self-map of $\mathbb{D}$ is inside the unit disk also leads to an eventual conjugacy. More generally, we have the following result. Although this lemma might be folklore, for the sake of completeness we give a short proof.

Lemma 2.9. *Let h be a holomorphic self-map of a hyperbolic domain $G \subset \hat{\mathbb{C}}$ such that h is not an automorphism of G and there is a fixed point p of h in G such that $\lambda := h'(p) \neq 0$. Then $|\lambda| < 1$ and $h \sim \lambda\mathrm{id}_{\mathbb{C}}$.*

Proof. By Montel's theorem, the family $\{h^n : n \in \mathbb{N}\}$ is normal. Since h is not an automorphism of G we conclude that no subsequence of $(h^n)_{n\in\mathbb{N}}$ converges to a non-constant limit function (see for instance [6, Theorem 7.2.4]). Hence $h^n \to p$ locally uniformly on G, which implies that $|\lambda| < 1$. Hence there is an open and connected neighbourhood V of p in G, a number $r > 0$, and a biholomorphic map $\phi : V \to D(0, r)$ such that $h(V) \subset V$, $\phi(p) = 0$, $\phi'(p) = 1$ and $\phi \circ h|_V = \lambda\phi$. It is easy to see that

$$\Phi(z) := \frac{1}{\lambda^n}\phi(h^n(z)) \quad \text{if} \quad h^n(z) \in V$$

is a well-defined holomorphic function on G such that $(\mathbb{C}, \lambda\mathrm{id}_{\mathbb{C}}, \Phi, V)$ forms an eventual conjugacy of h on G. □

Moreover, we make use of the following theorems due to P. Bonfert [11, Theorem 5.7 and Theorem 6.1].

Theorem 2.10 (Bonfert)**.** *Let h be a holomorphic self-map of a hyperbolic domain $G \subset \mathbb{C}$ without a fixed point, and without an isolated boundary fixed point, i.e. there is no isolated boundary point $a \in \partial G$ such that h extends holomorphically to a and fixes a. Suppose that $\lambda_G(h^n(z), h^{n+1}(z)) \to 0$ as $n \to \infty$ for some $z \in G$. Let $z_0 \in G$. Define*

$$\phi_n : G \to \mathbb{C}, z \mapsto \frac{h^n(z) - h^n(z_0)}{h^{n+1}(z_0) - h^n(z_0)}.$$

Then the sequence $(\phi_n)_{n\in\mathbb{N}}$ converges locally uniformly in G to a holomorphic function $\phi : G \to \mathbb{C}$ such that $\phi(h(z)) = \phi(z) + 1$ for all $z \in G$.

Theorem 2.11 (Bonfert)**.** *Let T be a Möbius transformation and $G \subset \mathbb{C}$ be a hyperbolic domain such that $T(G) \subset G$, $T(\infty) = \infty$, and T has no fixed point in G. Then*

$$\lambda_G(T^n(z), T^{n+1}(z)) \to 0 \quad \text{as} \quad n \to \infty \quad \text{for (any)} \quad z \in G$$

if and only if

$$\bigcup_{n\in\mathbb{N}} T^{-n}(G) = \mathbb{C} \quad \text{or} \quad \bigcup_{n\in\mathbb{N}} T^{-n}(G) = \mathbb{C} \setminus \{b\},$$

where $b \in \mathbb{C} \setminus G$ is a fixed point of T.