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0521682401 - Maths Trails: Working Systematically
Jennifer Piggott and Liz Pumfrey
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Problem solving and mathematical thinking

What is problem solving?

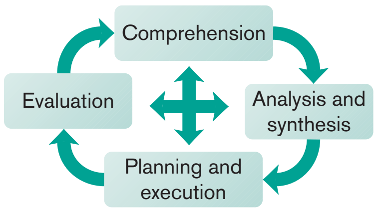
When we are presented with a mathematical problem, it is only a problem if we do not immediately know how to solve it. The process of problem solving is like a journey from a state of not knowing what to do, towards a destination which we hope will be the solution. The key is to have some strategies at our fingertips which will help us to identify a possible route through to a solution. Our mathematical journey is often full of twists and turns where we revisit ideas or need to step back and look for alternatives. Often a mistake or dead-end gives vital clues to the mathematics of the problem and is therefore crucial in the solution process.

To help us identify where we are during problem solving, or what might be a good strategy to try next, it is helpful to have a model. The diagram shows the problem-solving process as a cycle of activity, starting with comprehension and moving through analysis and synthesis towards planning, execution and evaluation. This cycle is not a neat process as we often need to revisit stages on the journey from problem posing to problem solving. Further details are given below.

Comprehension

- Making sense of the problem, retelling, creating a mental image
- Identifying all the relevant information

For example, we need to read the question carefully, re-read it several times and/or draw a picture to represent the problem in order to understand it.



Analysis and synthesis

- Identifying what is unknown and what needs finding
- Identifying and accessing prerequisite knowledge
- Conjecturing and hypothesising ('What if … ?')

Planning and execution

- Applying a model to what is known about the problem
- Thinking of ways of finding what is NOT known
- Planning the solution, which might include the consideration of novel approaches or strategies that have worked for similar problems
- Identifying possible mathematical knowledge and skills gaps that may need addressing
- Executing the solution, keeping track of what has been done
- Trying to make sense of any finding or progress made
- Posing new problems
- Communicating results

Evaluation

- Reflecting and reviewing of the solution
- Justifying conclusions
- Conducting a self-assessment concerning learning and the mathematical tools employed
- Thinking about other questions that could now be investigated

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When working with pupils this model can help us to talk about what they are doing and help them to set achievable and realistic goals. The model can act as a guideline for questioning or stimulus for discussion:

- 'Can you tell me what you think the problem is about?'
- 'What are you trying to find?'
- 'Have you seen anything like this before and what did you do then?'
- 'Could you have solved this in a different way?'

What is mathematical thinking?

The particular mathematical skills we need to use when problem solving are more than numeric, geometric and algebraic manipulation. They include ideas such as:

- modelling;
- visualising;
- working systematically;
- generalising.

We would class these skills as elements of mathematical thinking that are needed to engage in mathematical problem solving. This book focuses on the skills associated with *working systematically*.

The systematic trail

A trail is an organised set of curriculum resources, including teacher notes and pupil hints, designed to develop pupils' mathematical thinking.

We try to encourage a large range of mathematical thinking and problem-solving skills in our classrooms. However, it is not always so clear how to structure a programme of opportunities that introduce, develop and enhance particular skills. This trail places a particular emphasis on 'working systematically' and is designed to meet the needs of pupils between the ages of 10 and 14. The aim of this trail is to raise awareness of a possible toolkit of ideas that pupils (and teachers) can turn to and use in a range of contexts, rather than assuming one can achieve a mechanistic approach to work systematically. The trail has a sense of order and progression (see diagram on the next page), but individual problems can be used very effectively as an independent resource.

The trail consists of two complementary structures:

- It highlights different ways in which we expect pupils to be systematic.
- It provides opportunities for each of those skills to be developed.

There are no step-by-step rules that will always lead to a solution but, after working on this trail, students will be able to recognise where being systematic might be helpful and feel empowered to apply systematic approaches in those contexts.

Indications of prerequisite mathematical knowledge, links to standard curriculum documents and assessment guidelines are also included to help place the trail within current curriculum frameworks. This gives teachers the flexibility to look at particular problems as appropriate to their pupils' current needs.

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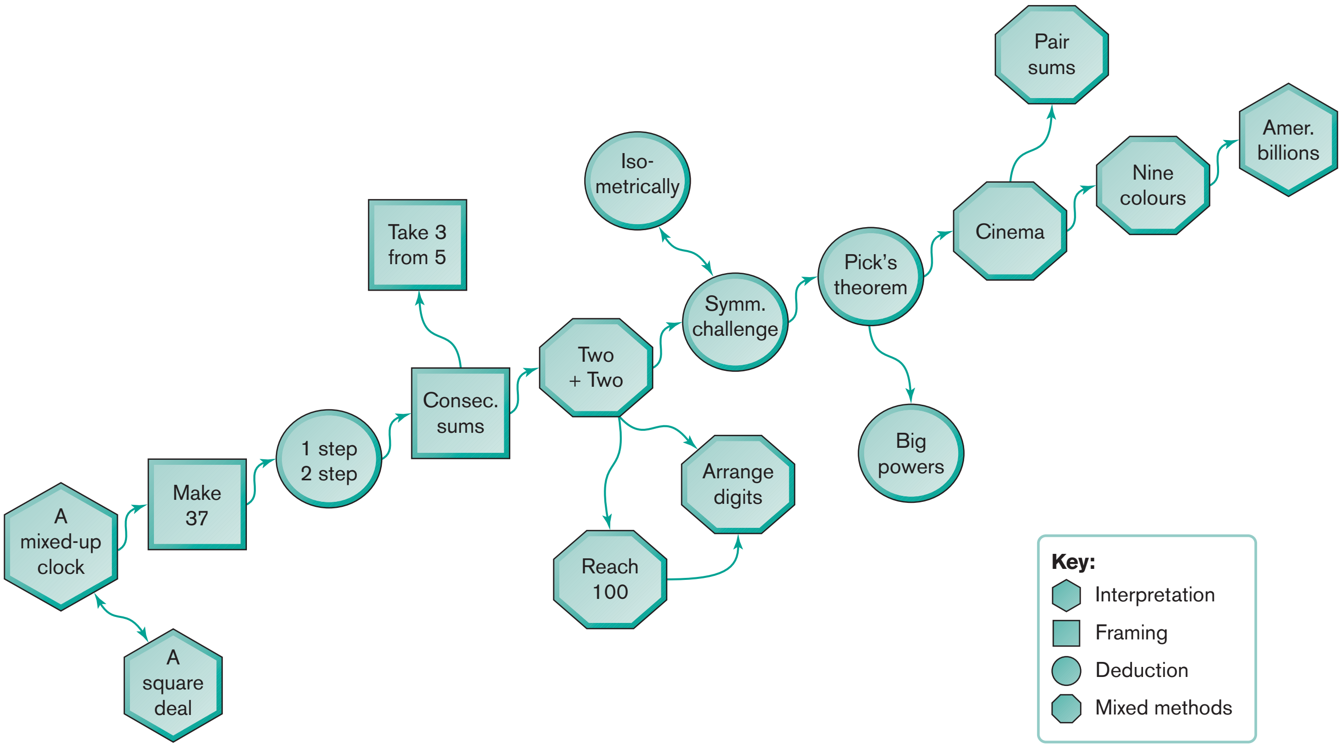
The systematic trail structure



The trail runs from bottom left to top right. In general, a problem which is further right and coded in the same shape requires higher proficiency in working systematically or involves some aspect of mathematics associated with a higher level of maturity.

Why use this systematic trail?

Pupils are often asked to 'go away and be systematic', but we are often not clear about what this means. This lack of clarity makes it difficult to discuss working systematically, in anything other than a superficial way, with our pupils. We cannot assume that someone can go away and, as if by magic, begin acting more systematically. There are skills we can identify that are about working systematically which can be discussed and shared with our pupils. This trail aims to support this dialogue.

In the context of problem solving, working systematically can be about working in an efficient and convincing way to solve certain groups of problems. Within this trail we have identified three main strands, or kinds, of ways in which you can 'work systematically'.

1 Interpretation: where data or information within a problem accessed in a systematic way enables entry into the solution. This is often represented by data which constrains the starting point of the activity.
   - 'A mixed-up clock' – this requires an organised and logical interpretation of the information to select the most helpful clues in the most helpful order.
   - 'A square deal' (similar to 'A mixed-up clock') – this requires the solver to use the clues in an ordered way to complete the square.
   - 'American billions' – there are lots of clues, some of which immediately fix some of the digits, but then further analysis of possibilities is required. A solution emerges as combinations are eliminated.

2 Framing: taking meaning from playing with the problem – recording outcomes which lead to identifying patterns that need to be validated. That is, the system helps you to identify a property you need to explain (the explanation may not be directly related to the initial recording process). It points you in a particular direction that might lead to a solution.

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- 'Make 37' – the recording identifies that the sum is even and the question then focuses on why this is the case.
- 'Consecutive sums' – the recording of sums of pairs of consecutive numbers, trios of consecutive numbers and so on yields patterns that then need explanation. For example, why is the sum of two consecutive numbers always odd?
- 'Take three from five' – this always appears to be possible, which then focuses the solver on why this seems to be the case.

3 Deduction: part of the explanation and underlying mathematics comes from the ability to plan and organise inputs to one or more algorithms by:

- Stepping up: starting with something simple and working logically through a sequence:
  - 'One step, two step' – starting with 1, 2, 3 steps, a pattern emerges which leads to a potential generalisation which, in turn, will need justifying.
- All possibilities: taking a logical approach which ensures coverage of all possibilities:
  - 'Symmetry challenge' – this requires an approach which ensures all possibilities are covered. Some patterns and observations made en route can be used to convince others that all solutions have been found.
  - 'Isometrically' – this is almost identical to 'Symmetry challenge', but with a slightly less familiar starting point.
- Simple to complex: using a system of recording, often starting with a series of simple cases, which throw direct light on a more complex problem (an extension of 'stepping up'):
  - 'Pick's theorem' – here the need to be systematic involves identifying separate parts (related to which variable to fix) of the problem then working through a series of connected cases.
  - 'Big powers' – here the separate parts are to look at powers of 3 and 4 separately and to identify a pattern which then needs to be applied to the more complex case. The solution requires several steps.

4 Mixed methods: some problems can be solved using a mixture of two or more of these strategies.

- 'Two and two' – the multiple solutions to this problem require an approach which is sure to consider all possibilities. The journey is supported by using observations of results to see patterns which can reduce the size of the problem and help the solver to home in on the range of possible solutions.
- 'Reach 100' – this and the next problem can either act as precursors or supplementary material to 'Two and two', requiring similar skills but being less demanding. The reason for putting them in the loop, and 'Two and two' into the main route of the trail, is simply that 'Two and two' is a rich enough problem on its own.
- 'Arrange the digits' – this continues the idea of 'Reach 100' and 'Two and two'.
- 'Cinema problem' – the above problems focus on a knowledge of place value and this problem uses similar skills in a different context (money). It is still necessary to work in a logical sequence which can be aided by the identification of key properties and patterns.
- 'Pair sums' – this extends the approach to another context.
- 'Nine colours' – this encourages strategies which, through experiment and observation, lead to the identification of patterns that can aid resolution. This is a 3-D problem which can prove more difficult for some problem solvers.

It is important to highlight the different features of working systematically to pupils so that they can begin to identify for themselves the sorts of strategies they are employing, or could employ, when solving problems.

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How to use the systematic trail

You might wish to use the trail as a 'course' for pupils over a short or long period of time, working as a whole class, in small groups or individually. The trail indicates an ordering of the materials to support each pupil's developing skills. However, it is also possible to dip in and out of the materials. The trail has a simple ordering of problems and is intended to illustrate aspects of working systematically arising from different contexts including number and space and shape. It is not intended to be a straitjacket. The timings indicated in the teacher notes for each problem are a guide, as the intention is to encourage extension and pupil investigation beyond what is made explicit.

The problems offer opportunities for pupils of a wide age and ability range, and do not imply a particular view of classroom organisation. However, there is an underlying message concerning classroom practice and the learning of mathematics as a collaborative experience, valuing the journey through a problem rather than just the answer. Whilst there is no need to offer group-work opportunities, there is an expectation that pupils will talk about their mathematical experiences en route as well as at the conclusion of their studies.

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Ideas for managing systematic sessions

The lesson notes included in this book are intended purely as a guide. As indicated in what follows, there are as many approaches to teaching as there are pupils in a class! All we can do is seed some ideas without any intention of being prescriptive.

The CD-ROM includes printable copies of problems in pdf format. All problems can be used as OHTs if appropriate. However, where a problem would benefit from slightly different layout as a whole-class teaching resource, we have produced a separate OHT for this purpose. These 'teaching' OHTs and any additional resources have been included on the CD-ROM.

Working with the whole class

The following might be a typical whole-class approach to a problem.

Introduce the problem

Give a brief description and perhaps start or model an approach. Then ask pupils to work entirely on their own for five minutes – just giving them time to play and familiarise themselves with the context and *think* about the problem. At this point it is worth emphasising that pupils are not expected to be working neatly towards a solution but simply finding out what the problem might be about; tell them that after a short time you will stop them and ask them to work with a partner (*pair*) for a further five minutes in order to talk about what they have discovered. The time spent working individually and in pairs is to identify initial ideas and to *share* them with the whole class. This part of the lesson corresponds to the 'comprehension' phase of the problem-solving process:

think–pair–share

The pupils can now start to work further on the problem based on their own and other pupils' findings.

Sharing and moving on

Stopping the groups at appropriate times as they make more sense of the problem offers opportunities for those who have not found a way in to perhaps get started and for others to refine and develop ideas as a community. Here it is not enough to 'know how to do it' but to leave room for new ideas and questions to be discussed. This is about valuing the journey, including the cul-de-sacs on the way or the different routes we may take. During this time pupils are beginning to identify potentially promising approaches to tackling the problem.

Planning and execution

After these early discussions pupils need further time to investigate the problem and consider possible routes to their solutions and/or consider further problems that might arise. Sharing and iteration of discussions will help to give all a sense of owning the mathematics and ensure that, as far as possible, many different approaches to the problem are considered, not simply 'the answer'.

Evaluation

During this last phase pupils discuss their findings, convincing themselves and their friends that any findings they have or conjectures that they wish to put forward are reasonable. Time considering different solutions and their 'efficiency' or 'accessibility' is invaluable in opening up the mathematics and helping pupils value different and elegant approaches.

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Working with individual pupils

Trails can also be a useful tool for teachers to use with individual pupils who need the challenge of problem-solving activities. Pupils that quickly grasp the mechanical aspects of mathematics but find it difficult to work on open-ended tasks can be encouraged to tackle some of the problems in the trail. The sense of direction and purpose of the trail, when shared with the pupil(s), can give them the opportunity to build up a repertoire of approaches to such problems, and give them more confidence when confronted with similar activities in the future.

Working with groups

Groups of pupils can often find their own way into problems simply by being given the context and being asked to:

- identify what the question is about;
- consider strategies for solution;
- plan what they are going to do;
- execute their plans.

One important aim might be to encourage pupils to communicate their findings to others in the class or their teacher, describing the problems they had as well as how they chose and executed their solutions. This communication does not have to take the form of written output but could be verbal or in the form of a poster or presentation.

Finally, encourage the group to answer evaluative questions such as:

- Could we have done this more efficiently?
- What have we learnt that is new?
- Have we met anything like this before and were we able to make connections?
- What additional questions did we come up with and answer whilst we were working on the problem?
- Are there some questions still to be answered?

Many further examples of problems are available on the NRICH website (www.nrich.maths.org) if you wish to extend any of the work in a particular area of mathematics, or simply to reinforce ideas and skills.

Prerequisite knowledge

There are two aspects to pupils' prerequisite knowledge that need to be considered. Firstly, we need to consider pupils' abilities to tackle problem-solving situations independently. This trail does not assume significant familiarity with applying problem-solving skills (in particular with working systematically) and has been designed for both the novice and experienced learner. As the main purpose of the trail is to support the development of problem-solving and mathematical thinking skills, this may mean that, as a teacher, the amount of scaffolding and support you will be offering will decrease as your pupils gain confidence through the trail.

Secondly, each problem depends upon knowledge of particular aspects of mathematical curriculum content which is detailed in the accompanying notes. For example, the trail starts with 'A mixed-up clock', which requires a knowledge of odd, even and consecutive numbers and mental arithmetic using the four operations. The problem itself consolidates these mathematical concepts.

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Assessment

Assessment for learning

The notes and other documentation for each problem aim to support formative and summative assessment opportunities. Solution notes to all the problems are included to give some guidelines on expected outcomes. Edited pupil solutions can be found on the NRICH website at www.nrich.maths.org. The advantage of looking at solutions on the website is that they will give you an idea of what to expect from your pupils.

The key outcome for all of the activities is to develop pupils' skills in working systematically in a range of contexts. If you are listing curriculum content objectives at the begining of the lesson, care should be taken not to close down the opportunities for pupils to be creative about the mathematics they use. For example, listing 'Recognise rotation symmetry' may restrict pupils' thinking. We therefore offer a cautionary note that the lesson objectives shared with the pupils should not reveal obvious routes to the solution.

A pupil self-assessment sheet is included on the CD-ROM and can be made available to pupils at the end of each problem. This encourages pupils to consider the following aspects of their work:

- **Independence**
  Did you manage to work through to a solution even though you might not have had a clear idea to start with?
  Did you need help from someone who knew what to do?
- **Working systematically**
  Did you manage to be organised in the way you worked on the problem? Can you describe any ways in which you were systematic?
- **Evaluation**
  If you look back on the problem, can you see other ways you could have tackled it which might, or might not, have been more effective and elegant?
- **New mathematics**
  What new mathematics have you learnt by doing this problem?
- **Communicating and justifying**
  Were you able to convince someone else that your solution was correct?
- **Curiosity**
  Can you give an example of anything in the problem that has fired your interest enough to look at ideas not included in the question itself?

Listening and questioning are important tools in the process of formative assessment. To support this:

- all problems have suggested prompts for teachers and mentors to use;
- pupils are encouraged to hypothesise and share ideas with fellow pupils, arguing their case – these are ideal opportunities to listen;
- whole-group discussions during the lesson can be used to reveal pupils' understanding, misconceptions or lack of awareness of the necessary mathematical knowledge;
- peer assessment can often shed valuable light on the understanding of the assessor as well as the assessed;
- reviewing and reflecting on the lesson outcomes with the pupils can help the teacher make judgements and also be used by pupils as an opportunity for self- or peer assessment.

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As highlighted earlier, much of the work and learning is about the journey through each problem. It is not necessary for pupils to have well-rounded, written solutions for sound assessment judgements to be made. Whilst feedback through marking is sometimes appropriate, oral and continuous feedback throughout the problem-solving process is just as valuable.

'To be effective feedback should cause thinking to take place'
(*Assessment for learning in everyday lessons*, DfES, 2004).

Assessing learning over the whole trail

This section considers the possible outcomes of completing the whole trail for assessment purposes. In particular this trail was designed to widen and deepen pupils' ability to work systematically in a range of contexts.

Pupils and teachers have the opportunity to assess what learning has taken place after each of the problems (see the pupil assessment sheet). Here are some concluding thoughts to help with the assessment process over the whole of the trail.

Pupils should look back over this trail and think about what they have been doing and therefore identify:

- ways in which they have been systematic;
- mathematical facts they have used;
- the range of mathematical skills they have employed;
- some thinking and problem-solving skills they have developed;
- instances where they were able to be independent and find things out for themselves;
- places where they have compared ideas or methods and evaluated their choice;
- times when they have asked themselves 'what if ...? what if not ...?' questions;
- other relevant questions to ask themselves and others;
- things that have fired their curiosity – and got them to ask more questions;
- where they have persevered and not been frightened by 'complicated work';
- places where they have found symbols and/or algebra useful;
- places where they have showed their ideas or findings to others.

Links to mathematics curriculum

The Key Stage 3 Framework for teaching mathematics states:

> 'Thinking skills underpin using and applying mathematics and the broad strands of problem solving, communication and reasoning. Well-chosen mathematical activities will develop pupils' thinking skills ... Used well, this approach can focus pupils' attention on the 'using and applying' or thinking skills that they have used so that they can apply these skills more generally in their mathematics work.'

All the problems in this trail will provide opportunities for pupils to develop their general problem-solving skills, whilst focusing more specifically on 'working systematically'. Where appropriate, explicit links to the 'Solving problems' strand of the Years 5 and 6 Framework, and the 'Using and applying mathematics to solve problems' strand of the Years 7 and 8 Framework are indicated in the table on the CD-ROM for each problem. The problems in the trail utilise and develop aspects of the standard mathematics curriculum, the 5–14 guidelines for Scotland and the NI Programmes of study. These links are also listed.

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A mixed-up clock

Interpretation

Prerequisite knowledge

- Understanding of number concepts such as odd, even and consecutive
- Mental arithmetic that enables pupils to undertake simple calculations using the four operations

Why do this problem?

This problem offers a good example of the need to be systematic as the solution can only be approached by systematically interpreting information given in the clues.

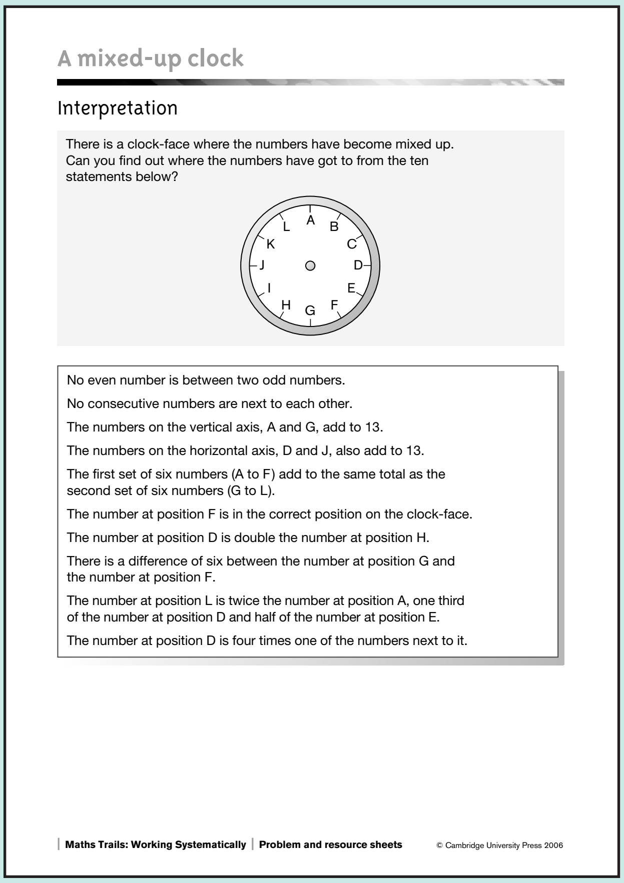
Time

Up to one lesson

Resources

CD-ROM: problem sheet, resource sheet

NRICH website (optional): www.nrich.maths.org, July 2002, 'A mixed-up clock'



A mixed-up clock

Interpretation

There is a clock-face where the numbers have become mixed up. Can you find out where the numbers have got to from the ten statements below?

No even number is between two odd numbers.

No consecutive numbers are next to each other.

The numbers on the vertical axis, A and G, add to 13.

The numbers on the horizontal axis, D and J, also add to 13.

The first set of six numbers (A to F) add to the same total as the second set of six numbers (G to L).

The number at position F is in the correct position on the clock-face.

The number at position D is double the number at position H.

There is a difference of six between the number at position G and the number at position F.

The number at position L is twice the number at position A, one third of the number at position D and half of the number at position E.

The number at position D is four times one of the numbers next to it.

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Introducing the problem

Begin the lesson with some mental warm-ups which remind the class of number properties such as odd, even and consecutive. For example, invite pupils to suggest:

- three consecutive odd numbers greater than 50 but less than 70;
- two numbers whose sum is an even number less than 100;
- two numbers whose product is an odd number less than 100.

Other questions:

- How many different sets of three consecutive odd numbers are there between 50 and 70?
- What are the largest and smallest totals you can make using three consecutive odd numbers between 50 and 70?

Show the resource sheet and briefly describe what the problem entails, without reading the text out. Ask pupils to suggest how they might begin to tackle the problem. Reading the information given might be the most appropriate thing to do, and some pupils may have already begun!

- Is this like an ordinary mathematics exercise? Can you start at the top and work your way through to the end?

Main part of the lesson

Read through the first few clues given underneath the clock-face. Stop after each and discuss with the pupils what exactly can be learnt from it. Some clues might lend themselves to writing equations if pupils are happy with this and others may be redundant at this stage. It might be important to emphasise that clues which appear useless at first may prove useful later. Encourage pupils to keep track of what they have learnt and what they have, or have not, used.

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Suggest that the pupils look through each clue and decide what can be filled in immediately. It will become obvious that the sixth clue is immediately helpful and discussion could ensue about what the pupils might do next (carry on checking down the list) and what to do after that (go back to the top!).

Ask the pupils to complete the clock-face, either individually or in pairs. It would be appropriate to ask them to suggest ways of recording their work.

It may be useful to stop occasionally to discuss progress and strategies with the class.

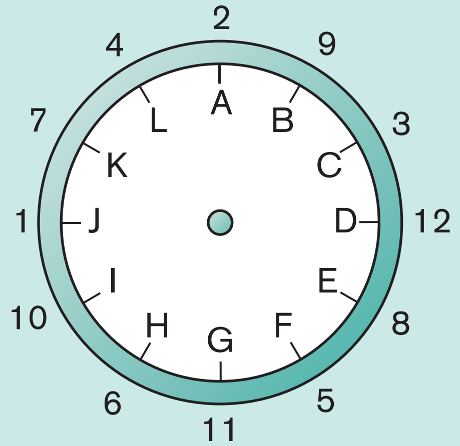
Plenary

A suitable plenary would consist of requesting pairs or individuals to explain the steps they took to solve the problem. Prompts might include:

- Is this the only clue that is useful now? Why?
- What other information will we need before being able to use this clue?
- Are any of the clues completely redundant?
- How do you think this problem was created?

It will be interesting to discuss with the pupils the point at which there is an element of choice in the order the clues are utilised. Making it explicit that this ordering of the information constitutes a *systematic approach* will help pupils begin to understand a meaning of being systematic so that they can apply this in future problems.

Solution notes



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