Cambridge University Press
0521682398 - Maths Trails: Generalising
Jennifer Piggott and Liz Pumfrey
Excerpt
More information

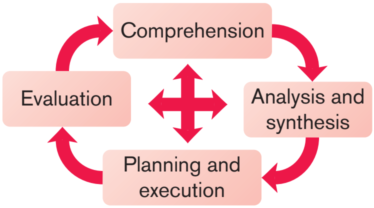
do not immediately know how to solve it. The process of problem solving is like a journey from a state of not knowing what to do, towards a destination which we hope will be the solution. The key is to have some strategies at our fingertips which will help us to identify a possible route through to a solution. Our mathematical journey is often full of twists and turns where we revisit ideas or need to step back and look for alternatives. Often a mistake or dead-end gives vital clues to the mathematics of the problem and is therefore crucial in the solution process.

To help us identify where we are during problem solving, or what might be a good strategy to try next, it is useful to have a sense of direction. A model for problem solving can help with this. There are many such models but the one below is a good starting point. Although it is written in a linear sequence of activity the reality is a cyclic process often revisiting places on the journey from problem posing to problem solving (see diagram).

Comprehension

- Making sense of the problem, retelling, creating a mental image
- Identifying all the relevant information

For example, we need to read the question carefully, re-read it several times and/or draw a picture of the problem in order to understand it.



Analysis and synthesis

- Identifying what is unknown and what needs finding
- Identifying and accessing required prerequisite knowledge
- Conjecturing and hypothesising ('What if ... ?')

Planning and execution

- Applying a model to what is known about the problem
- Thinking of ways of finding what is NOT known
- Planning the solution, which might include the consideration of novel approaches or strategies that have worked for similar problems
- Identifying possible mathematical knowledge and skills gaps that may need addressing
- Executing the solution, keeping track of what has been done
- Trying to make sense of any progress made
- Posing new problems
- Communicating results

Evaluation

- Reflection and review of the solution
- Justifying conclusions
- Self-assessment concerning learning and mathematical tools employed
- Thinking about other questions that could now be investigated

- 'Could you have solved this in a different way?'

What is mathematical thinking?

The particular mathematical skills we need to use when problem solving are more than numeric, geometric and algebraic manipulation. They include ideas such as:

- modelling;
- visualising;
- being systematic;
- generalising.

We would class these skills as elements of mathematical thinking that are needed to engage in mathematical problem solving. This book focuses on the skills associated with *generalising*.

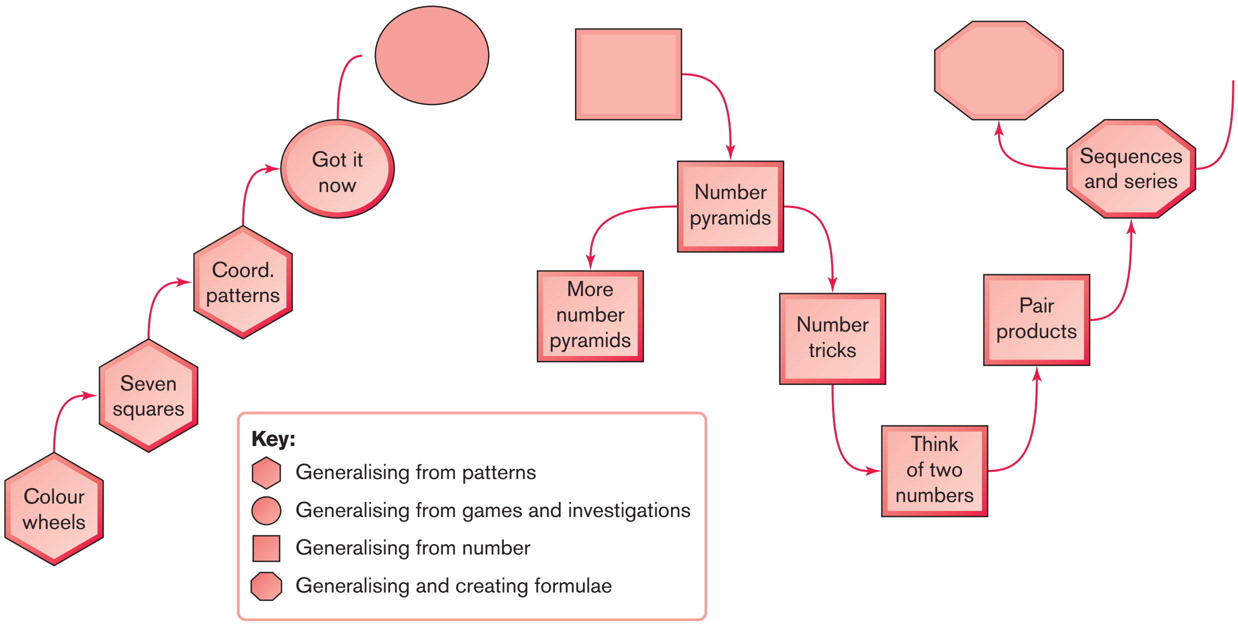
The generalising trail

A trail is an organised set of curriculum resources, including teacher notes and pupil hints, designed to develop pupils' mathematical thinking.

There is a large range of mathematical thinking and problem-solving skills we try to encourage in our classrooms. However, it is not so clear how to structure a programme of opportunities that introduce, develop and enhance particular skills. This trail places a particular emphasis on 'generalising' and is designed to meet the needs of pupils between the ages of 10 and 14. The aim of this trail is to encourage pupils to generalise from a variety of problem-solving settings that do not rely heavily on the need for algebraic manipulation. In many standard curriculum documents it is assumed that generalising requires the use of algebra and therefore it is implied that young pupils cannot generalise. This trail is based upon the view that pupils of all ages can generalise and encourages the communication of any generalisation in an appropriate form (which need not involve algebra).

The trail's sense of order and progression (see the diagram on the next page) develops pupils' generalising skills in a systematic way. The aim of the trail is to raise awareness of a toolkit of ideas that pupils (and teachers) can turn to and use in a range of problem-solving contexts. The trail's structure is designed to equip pupils with this valuable toolkit. There is no such thing as a mechanistic approach to 'generalising'. There are no step-by-step rules that will always lead to a solution but, after working on this trail, pupils should feel more confident to generalise mathematically.

Indications of prerequisite mathematical knowledge, links to standard curriculum documents and assessment guidelines are also included to help place the trail within current curriculum frameworks. This gives teachers the flexibility to look at particular problems as appropriate to their pupils' current needs.



The trail runs from bottom left to top right. In general a problem which is further right and coded with the same shape requires higher proficiency in generalising or involves some aspect of mathematics associated with a higher level of maturity.

Why use this generalising trail?

Pupils may have already met mathematical ideas such as:

- factors and multiples;
- coordinate systems;
- areas of squares;
- Pythagoras' theorem.

They may have used these ideas in different places and at different times in school or at home. In doing so, they will often have had to apply some generality. That is, they will have taken the underlying principle of an idea and applied it in contexts frequently far removed from their original experience.

On such journeys pupils may have searched and perhaps found repeating patterns, perhaps noticed similar features and perhaps spotted a distinct level of sameness about what they do. The subsequent expression of any sameness is the first step towards generality.

Often pupils only experience the applying of formulae and do not experience the satisfaction of understanding them sufficiently to feel confident about applying them in unfamiliar contexts. This deeper understanding reflects the fuller process of being able to generalise ideas and it comes from exploring, applying and explaining the underpinning mathematics.

This resource is an attempt to develop an understanding of this underlying but important journey to generalising in mathematics.

- Generalising is an important process in doing mathematics.
- Generality lies at the heart of mathematics.

thoughts about the amount of sameness within a context will be formulated and expressed, and possibly refined as further discussion ensues. The hope is that pupils will conjecture about what they see and test out all their observations.

Conjecturing is not always easy, mistakes can be made, and misconceptions can arise. But these incidences can also provide stimulating distractions on the journey to understanding and help strengthen foundations.

How to use the generalising trail

You might wish to use the trail as a 'course' for pupils over a short or long period of time, working as a whole class, or in small groups or individually. The trail indicates an ordering of the materials to support each pupil's developing skills. However, it is also possible to dip in and out of the materials. The trail has a simple ordering of problems. Several of the problems have a number of similar activities and extensions contained within them but two of them ('Number pyramids' and 'Sequences and series') have a distinct but connected extension activity shown separately on the map above. The trail is intended to illustrate aspects of generalising arising from different types of contexts including patterns, number and games as well as a developmental pathway for improving generalising skills. It is not intended to be a straitjacket. The timings indicated in the teacher notes for each problem are a guide as the intention is to encourage extension and pupil investigation beyond what is made explicit.

The problems offer opportunities for pupils of a wide age and ability range, and do not imply a particular view of classroom organisation. However, there is an underlying message concerning classroom practice and the learning of mathematics as a collaborative experience, valuing the journey through a problem rather than just the answer. While there is no need to offer group-work opportunities there is an underpinning expectation that pupils will be given opportunities to talk about their mathematical experiences *en route* as well as at the 'conclusion' of their studies.

Ideas for managing generalising sessions

The lesson notes included in this book are intended purely as a guide. As indicated in what follows, there are as many approaches to teaching as there are pupils in a class! All we can do is seed some ideas without any intention of being prescriptive.

The CD-ROM includes printable copies of problems in pdf format. All problems can be used as OHTs (overhead transparencies) if appropriate. However, where a problem would benefit from a slightly different layout as a whole-class teaching resource, we have produced a separate OHT for this purpose. These 'teaching' OHTs and any additional resources are on the CD-ROM.

play and familiarise themselves with the context. At this point it is worth emphasising that pupils are not expected to be working neatly towards a solution but simply finding out what the problem might be about; tell them that after a short time you will stop them and ask them to work with a partner for a further five minutes in order to share what they have discovered. The time spent working individually and in pairs is to identify and share initial ideas that can then be discussed as a whole class. This part of the lesson corresponds to the 'comprehension' phase of the problem-solving process.

Sharing and moving on

Stopping the groups after a further short period of time to share findings and ideas of what the problem is about, offers opportunities for those who have not found a way into the problem to perhaps 'get started' and for the chance to refine and develop ideas as a community. Here it is not enough to 'know how to do it' but to leave room for new ideas and questions to be discussed. This is about valuing the journey, including the cul-de-sacs we may take on the way or the different routes we may take. During this time pupils are beginning to analyse and synthesise the problem.

Planning and execution

After these early discussions pupils need time to investigate the problem and consider possible routes to their solutions. Sharing and iteration of discussions will help to give all a sense of owning the mathematics and ensure that, as far as possible, many different approaches to the problem are considered, not simply 'the answer'.

Evaluation

During this last phase pupils discuss their findings, convincing themselves and their friends that any findings they have or conjectures that they wish to put forward are reasonable. Time spent considering different solutions and their 'efficiency' or 'accessibility' is invaluable in opening up the mathematics and helping pupils to value different approaches.

Working with individual pupils

Trails can also be a useful tool for teachers to use with individual pupils who need the challenge of problem-solving activities. Pupils that quickly grasp the mechanical aspects of mathematics but find it difficult to work on open-ended tasks can be encouraged to tackle some of the problems in the trail. The sense of direction and purpose of the trail, when shared with the pupil(s), can give them the opportunity to build up a repertoire of approaches to such problems, and give them more confidence when confronted with similar activities in the future.

- plan what they are going to do;
- execute their plans.

One important aim might be to encourage pupils to communicate their findings to others in the class or their teacher, describing the problems they had as well as how they chose their approach and executed their solutions. This communication does not have to take the form of written output but could be verbal or in the form of a poster or presentation.

Finally, encourage the group to answer evaluative questions such as:

- Could we have done this more efficiently?
- What have we learnt that is new?
- Have we met anything like this before and were we able to make connections?
- What additional questions did we come up with and answer while we were working on the problem?
- Are there some questions still to be answered?

Many further examples of problems are available on the NRICH website (www.nrich.maths.org) if you wish to extend any of the work in a particular area of mathematics, or simply to reinforce ideas and skills.

Prerequisite knowledge

There are two aspects to pupils' prerequisite knowledge that need to be considered. Firstly, we need to consider pupils' abilities to tackle problem-solving situations independently. This trail does not assume significant familiarity with applying problem-solving skills (in particular with generalising) and has been designed for both the novice and the experienced learner. As the main purpose of the trail is to support the development of problem solving and mathematical thinking skills, this may mean that, as a teacher, the amount of scaffolding and support you will be offering will decrease as your pupils gain in confidence through the trail.

Secondly, each problem depends upon knowledge of particular aspects of mathematical curriculum content which is detailed in the accompanying notes. For example, the trail starts with 'Colour wheels', which requires some basic knowledge of factors and multiples, and can be used to extend and develop these mathematical concepts.

Assessment

Assessment for learning

The notes and other documentation for each problem aim to support formative and summative assessment opportunities. Sample solutions to all the problems are included to give some guidelines. Where these have been written algebraically it is simply for the purpose of being concise; it is not intended to suggest that the use of algebra is an expected outcome. Edited pupil solutions can be found on the NRICH website (www.nrich.maths.org). The advantage of looking at solutions on the website is that they will give you an idea of what to expect from your pupils.

A pupil self-assessment sheet is included on the CD-ROM and can be made available to pupils at the end of each problem. This encourages pupils to consider the following aspects:

- **Independence**
  Did you manage to work through to a solution even though you might not have had a clear idea to start with?
  Did you need help from someone who knew what to do?
- **Generalising**
  Did you manage to explain or communicate a pattern that you saw, or idea you had, to someone else?
- **Evaluation**
  If you look back on the problem, can you see other ways you could have tackled it which might, or might not, have been more effective and elegant?
- **New mathematics**
  What new mathematics have you learned by doing this problem?
- **Communicating and justifying**
  Were you able to convince someone else that your solution was correct?
- **Curiosity**
  Can you give an example of anything in the problem that has fired your interest enough to look at ideas not included in the question itself?

Listening and questioning are important tools in the process of formative assessment. To support this:

- all problems have suggested prompts for teachers and mentors to use;
- pupils are encouraged to hypothesise and share ideas with fellow pupils, arguing their case – these are ideal opportunities to listen;
- whole group discussions during the lesson can be used to reveal pupils' understanding, misconceptions or lack of awareness of the necessary mathematical knowledge;
- peer assessment can often shed valuable light on the understanding of the assessor as well as the assessed;
- reviewing and reflecting on the lesson outcomes with pupils can help the teacher make judgements and also be used by pupils as an opportunity for self or peer assessment.

As highlighted earlier, much of the work and learning is about the journey through each problem. It is not necessary for pupils to have well-rounded, written solutions for sound assessment judgements to be made. While feedback through marking is sometimes appropriate, oral and continuous feedback throughout the problem-solving process is just as valuable.

'To be effective feedback should cause thinking to take place'
(*Assessment for learning in everyday lessons*, DfES, 2004).

- games and investigations.

Pupils and teachers have the opportunity to assess what learning has taken place after each of the problems (see pupil assessment sheet). Here are some concluding points to help with the assessment process over the whole of the trail.

Pupils should look back over this trail and think about what they have been doing and therefore identify:

- mathematical facts they have used;
- the range of mathematical skills they have employed;
- some thinking and problem-solving skills they have developed;
- instances where they were able to be independent – and find things out for themselves;
- places where they have compared ideas or methods and evaluated their choice;
- times when they have asked themselves 'what if ... ? what if not ... ?' questions;
- other relevant questions to ask themselves and others;
- things that have fired their curiosity – and got them to ask more questions;
- where they have persevered and not been frightened by 'complicated work';
- places where they have found symbols and/or algebra useful.

Links to the mathematics curriculum

The Key Stage 3 Framework for teaching mathematics states:

> 'Thinking skills underpin using and applying mathematics and the broad strands of problem solving, communication and reasoning. Well-chosen mathematical activities will develop pupils' thinking skills ... Used well, this approach can focus pupils' attention on the 'using and applying' or thinking skills that they have used so that they can apply these skills more generally in their mathematics work.'

All the problems in this trail will provide opportunities for pupils to develop their general problem-solving skills, while focusing more specifically on 'generalising'. Where appropriate, explicit links to the 'Solving problems' strand of the Years 5 and 6 Framework, and the 'Using and applying mathematics to solve problems' strand of the Years 7, 8 and 9 Framework are indicated in the table below for each problem. The problems in the trail also utilise and develop aspects of the 'standard' mathematics curriculum, the 5–14 guidelines for Scotland and the NI Programmes of study. These links are also listed.

Cambridge University Press
0521682398 - Maths Trails: Generalising
Jennifer Piggott and Liz Pumfrey
Excerpt
More information

Problem	Solving problems/Using and applying	Curriculum content	Links to 5–14 guidelines	Links to the NI programmes of study
Colour wheels	Recognise and explain patterns and relationships (Y5/6: 79) Explain a generalised relationship in words (Y5: 80–81) Explain and justify methods and conclusions orally and in writing (Y7: 30–31)	Recognise and extend number sequences (Y5/6: 17) Recognise multiples (Y5/6: 19)	Work with patterns and relationships within and among multiplication tables. (NMM Level C)	Developing processes in mathematics (KS2/3) Patterns, Relationships and Sequences (KS2 a, b) Understanding Number and Number Notation (KS3 a) Patterns, Relationships, Sequences and Generalisations (KS3 a)
Seven squares	Explain a generalised relationship in words (Y5: 81) Suggest extensions to problems, conjecture and generalise; identify exceptional cases or counter-examples (Y7: 32) Explain and justify methods and conclusions orally and in writing (Y7: 30–31)	Mental calculation strategies (Y5: 40–47, 60–65)	Work with patterns and relationships by adding or taking something (NMM Level C)	Developing processes in mathematics (KS2/3) Operations and their Applications (KS2 b)
Coordinate patterns	Present and interpret solutions (Y7: 30–31) Suggest extensions to problems (Y7: 32–35) Explain and justify methods and conclusions orally and in writing (Y7: 30–31)	Coordinates in all four quadrants (Y6: 109, Y7: 218–219) Sequences, functions and graphs (Y7: 144–177) Understand negative numbers as positions on a number line; order, add and subtract positive and negative numbers in context (Y7: 48–51)	Discuss position and movement – use a coordinate system to locate a point on a grid (SPM Level D), in all four quadrants (SPM Level E) Continue and describe more complex sequences (NMM Level D) Add and subtract positive and negative numbers in applications (NMM Level E)	Developing process in mathematics (KS2/3) Patterns, Relationships and Sequences (KS2 a, c) Measures (KS2 g) Number Operations and Applications (KS3 d) Patterns, Relationships, Sequences and Generalisations (KS3 a) Position, Measurement and Direction (KS3 b)
Got it now	Recognise and explain patterns and relationships, generalise and predict (Y5/6: 79) Conjecture and generalise (Y8/9: 32–35) Justify generalisations, arguments or solutions (Y9: 32–35)	Recognise multiples (Y5/6: 19) Consolidate the rapid recall of number facts (Y7: 88–91)	Continue and describe more complex sequences (NMM Level D) Add and subtract mentally (NMM Level C)	Developing processes in mathematics (KS2/3) Operations and their Applications (KS2 a) Understanding Number and Number Notation (KS3 a) Number Operations and Applications (KS3 a)