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Permutation Groups and Cartesian Decompositions

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To John, James, and Tim

To Jutka, Benjamin, and Alice

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Preface

This is a book about permutation groups, their fundamental theory and applications. It focuses on those permutation groups which are most useful for studying symmetric structures. We hope the book will be enjoyed and used by many students and researchers with an interest in mathematical symmetry.

We give a modern treatment not only of primitive permutation groups but also of the larger families of quasiprimitive and innately transitive groups. These groups occur naturally in applications where reductions to primitive groups are not available. For example, an appropriate quotient operation, applied to a structure with particular symmetry properties, often yields a smaller structure with the same symmetry properties and admitting a subgroup of automorphisms that in addition is quasiprimitive. The quasiprimitive group so obtained is not in general the full automorphism group. Thus knowing the possible overgroups of a given quasiprimitive group helps us understand possible additional symmetries of this smaller structure.

Preserving a cartesian decomposition of the underlying point set turns out to be an especially important property that in some cases determines the ‘type’ of a primitive, or quasiprimitive, or innately transitive permutation group. Thus, when describing the overgroups of a given primitive or quasiprimitive or innately transitive group, a fundamental problem we need to solve is the determination of all the cartesian decompositions it preserves. This book is the first to provide a mathematical framework to attack this problem. It treats, as far as possible, general permutation groups – not necessarily finite ones. For finite groups, it provides surprisingly complete solutions, making use of detailed information about the finite simple groups, based on the finite simple group classification.

The systematic study of the overgroups of a given finite permutation

group was initiated by Cheryl's 1990 paper that characterised the overgroups of finite primitive permutation groups. For the applications of the O'Nan–Scott Theorem to the study of symmetric graphs, this theory needed to be extended to finite quasiprimitive permutation groups and Cheryl embarked on this research project from the beginning of the 1990s. The possibilities for finite quasiprimitive groups are far greater than for finite primitive groups, and it became apparent that describing the overgroups of a quasiprimitive group that are wreath products in product action was in itself a significant research project. For a quasiprimitive permutation group, the existence of such an overgroup is equivalent to preserving a homogeneous cartesian decomposition.

The origins of the research program concerning the overgroups of quasiprimitive permutation groups can be traced back to a conversation between Robert Baddeley and Cheryl during a conference at Oberwolfach in the 1990s. The project began as a collaboration between Robert and Cheryl. Both of them were profoundly influenced by discussions with L. (Laci) G. Kovács regarding the scope and objectives of their joint work. At the end of the 90s, Robert left academia and Cheryl completed several papers that presented the findings of their decade-long research into this problem. However, the work to describe the overgroups of finite quasiprimitive permutation groups that are wreath products in product action was left unfinished, and hence unpublished. Robert left Cheryl a substantial manuscript that contained an account of their joint work up to then. When Csaba started his post-doctoral research position at The University of Western Australia in September of 1999, Cheryl and he decided that further developing the ideas laid out in Robert's manuscript was an appropriate and worthwhile project. Thus the work was developed and completed by Csaba and Cheryl. The results inevitably spread across multiple journal articles and we hope that presenting them in this book will make them more accessible.

When the first draft of our book was essentially complete, we became aware that many of the questions we had addressed for finite permutation groups were important also for infinite permutation groups. In particular, discussions with Simon Smith convinced us that the book should be written, as far as possible, in the context of general groups – and that in particular the theory of cartesian decompositions should be developed for families of infinite permutation groups. The theory of cartesian decompositions is given for finite rank decompositions, and restriction to finite groups is made only when we wish to use the powerful information available from the finite simple group classification.

We are grateful to many people for helping us complete this research project and for making this book possible. In the first place, our thanks go to Robert W. Baddeley whose manuscript was an important milestone heavily influencing the early drafts of this book. We are grateful to Simon Smith for persuading us to present the material, whenever possible, without restricting to finite groups. We thank Roger Astley and Clare Dennison of Cambridge University Press for their infinite patience and encouragement.