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Modular Representations of Finite Groups of Lie Type

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Preface

"I confess I like to take account of possibilities. Don't you know mathematics are my hobby? Did you ever study algebra? I always have an eye on the unknown quantity."

Henry James, *The Story of a Year*

Ich predige die Mathematik . . . . . . Die Beschäftigung mit der Mathematik, sage ich, ist das beste Mittel gegen die Kupidität. Staatsanwalt Paravant, der stark angefochten war, hat sich drauf geworfen, er hat es jetzt mit der Quadratur des Kreises und spürt große Erleichterung.

Thomas Mann, *Der Zauberberg*

– Ah! c'etait impossible, les cours duraient parfois fort tard.
– Même après 2 heures du matin? demandait le baron.
– Des fois.
– Mais l'algèbre s'apprend aussi facilement dans un livre.
– Même plus facilement, car je ne comprends pas grand'chose aux cours.
– Alors? D'ailleurs l'algèbre ne peut te servir à rien.
– J'aime bien cela. Ca dissipe ma neurasthénie.

Marcel Proust, *Sodome et Gomorrhe*

Whatever its therapeutic value may be, group theory offers plenty of diversions and challenges. In particular, the study of finite groups of Lie type and their representations (ordinary or modular) leads to deep questions, many of which are still unsolved.

A more accurate but cumbersome title for this book would be: "A guide to the modular representation theory of finite groups of Lie type in the defining characteristic p". As a subtheme, the relationship between ordinary and modular representations is explored, in the context of Deligne–Lusztig characters. However, I have stopped short of treating the "cross-characteristic" case, where representations are studied over fields of prime characteristic ℓ different from p (see the recent monograph by Cabanes and Enguehard [**78**]). In that case the questions and methods are quite different, being less connected with algebraic group representations and more intertwined with Deligne–Lusztig theory.

The approach here emphasizes the remarkably close parallels with the representation theories of the ambient simple algebraic group and its Frobenius kernels. Roughly speaking, the key representations of our finite group over a field of p^r elements closely resemble those of the rth Frobenius kernel (the case $r = 1$ corresponding to the Lie algebra of the algebraic group).

The treatment here is necessarily far from self-contained. It relies heavily on the theory exposed in Jantzen's book *Representations of Algebraic Groups*, whose second edition is referred to here as [**RAGS**]. However, the main prerequisites for study of the finite groups are concentrated in just the core chapters of Jantzen's Part II. (These remain largely unchanged from the first edition, though the coverage of Lusztig's Conjecture is significantly expanded in the new edition.)

One of my goals has been to make the subject more accessible to those working in neighboring parts of group theory, number theory, and topology. With this in mind, I have adopted some basic notational conventions at the outset in 1.8 and then tried to keep the chapters relatively independent. When it seems most useful I have given proofs in detail, but in the later chapters I have opted for informal exposition accompanied by examples and precise references.

The literature of this subject extends in many directions and often deals with special cases not yet possible to treat in a general framework. Even the large reference list I have assembled here is incomplete, though I have tried to include the primary sources and to attribute results correctly.

It is common in the theory of groups of Lie type to find "generic" patterns that are independent of p when p is sufficiently large. Partly for this reason, I have confined the treatment at first to Chevalley groups and their twisted analogues in types A, D, E_6. A concluding chapter summarizes what can be said about the groups of Suzuki and Ree, which are defined only in characteristic 2 or 3.

Here is a quick overview of the main topics discussed:

- Standard characterizations of the finite groups of Lie type
- Simple modules, from the classical results of Curtis and Steinberg to Lusztig's Conjecture
- Blocks, projective modules, and Cartan invariants
- Extensions of simple modules, or more generally Loewy series of projectives

- Cohomology of the finite groups relative to algebraic group cohomology, leading to the study of support varieties
- Decomposition behavior of ordinary characters (and Deligne–Lusztig characters) modulo p
- Special features of groups of type G_2, finite general linear groups, and the groups of Suzuki and Ree

Notation requires special attention, since there are parallel developments involving algebraic groups and finite groups. Various compromises are made in the literature, none entirely satisfactory. My own compromise is to use boldface letters such as $\mathbf{G}$ for algebraic groups (whereas on a chalkboard one might underline or use script letters) and roman letters such as G for their finite counterparts. Typically $G = \mathbf{G}^F$, the fixed points under a Frobenius map F.

I am grateful to many colleagues who have provided valuable feedback on earlier drafts of the book. Among these must be singled out Jens Carsten Jantzen, whose contributions to the subject have been of fundamental importance.