CHAPTER 1

Finite Groups of Lie Type

We begin with a brief review of the standard ways in which finite groups of Lie type are classified, constructed, and described. One complication is the multiplicity of approaches to describing this family of groups, which leads in turn to differing notational conventions in the literature. Our viewpoint will be mainly that of algebraic groups over finite fields (1.1), reformulated in terms of Frobenius maps (1.3). But occasional use will be made of the convenient axiomatic approach afforded by BN-pairs: see 1.7 and Chapter 7 below.

Even within the framework of algebraic groups, there is more than one way to organize the finite groups. Steinberg's unified description of the groups as fixed points of endomorphisms of algebraic groups is undoubtedly the most elegant and useful. However, in our treatment of modular representations it will be convenient to keep the groups of Ree and Suzuki (defined only in characteristic 2 or 3) largely separate from the other groups: see Chapter 20. These groups arise less directly from the ambient algebraic groups and of course do not exhibit any "generic" behavior for large primes p as other groups of Lie type do.

To conclude this introductory chapter we establish in 1.8 some standard notation.

1.1. Algebraic Groups over Finite Fields

The finite groups of Lie type are close relatives of the groups $\mathbf{G}(k)$ of rational points of algebraic groups defined over a finite field k. Here $\mathbf{G}$ is an affine variety with group operations given by regular functions, identified with its points over an algebraically closed field K. When the subfield k is perfect, $\mathbf{G}$ is *defined over* k precisely when it is the common set of zeros of a family of polynomials with coefficients in k. A standard example is the special linear group $\mathrm{SL}(n, K)$, which is defined over the prime field in K.

Here we recall some essential facts, referring to several textbooks for details: Borel [**55**, §16], Humphreys [**220**, §34–35], Springer [**385**]. Using a basic theorem of Lang (see 1.4 below), one shows without too much difficulty:

THEOREM. *Let k be a finite field having $q = p^r$ elements and let $\mathbf{G}$ be any connected algebraic group defined over k. Then:*

(a) $\mathbf{G}$ *is **quasisplit**, meaning it has a Borel subgroup defined over k. Moreover, all such Borel subgroups are conjugate under $\mathbf{G}(k)$.*
(b) $\mathbf{G}$ *has a maximal torus* $\mathbf{T}$ *defined over k, which lies in a Borel subgroup* $\mathbf{B}$ *defined over k. Here $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$, with* $\mathbf{U}$ *the unipotent radical of* $\mathbf{B}$.
(c) *Given an **isogeny** φ (an epimorphism with finite kernel) from* $\mathbf{G}$ *onto another connected algebraic group* $\mathbf{H}$ *over k, the finite groups $\mathbf{G}(k)$ and*

$\mathbf{H}(k)$ *have the same order even though the map* $\mathbf{G}(k) \to \mathbf{H}(k)$ *induced by* φ *need not be surjective.*

While the proof of part (c) involves some algebraic geometry, the coincidence of orders can be verified directly in special cases. Consider for example the isogeny $\mathrm{SL}(n,K) \to \mathrm{PGL}(n,K)$ obtained by restricting the natural map $\mathrm{GL}(n,K) \to \mathrm{GL}(n,K)/K^\times = \mathrm{PGL}(n,K)$ (where $K^\times$ is identified with scalar matrices). Taking $G = \mathrm{GL}(n,k)$, note that $\det : G \to k^\times$ is surjective and has kernel $\mathrm{SL}(n,k)$, while the surjective map $G \to G/k^\times = \mathrm{PGL}(n,k)$ has kernel $k^\times$. Thus $|SL(n,k)| = |PGL(n,k)|$.

Our main concern is with connected groups, though some nonconnected ones (such as normalizers of maximal tori or centralizers of arbitrary elements in reductive groups) also occur naturally. While any finite linear group over a finite field may be regarded as an algebraic group, the Borel–Tits structure theory essentially relies on connectedness assumptions.

We take as background the standard theory exposed in the texts mentioned above (and used heavily by Jantzen in [**RAGS**]). This focuses on a connected reductive group $\mathbf{G}$ such as $\mathrm{GL}(n,K)$, whose derived group is connected and semisimple. In turn, such a semisimple group decomposes as the almost-direct product of simple algebraic groups (having no proper connected normal algebraic subgroups). Chevalley's classification of these simple groups shows that they fall into essentially the same families over K as over $\mathbb{C}$.

Each simple algebraic group has a Lie type A–G (indexed by the **rank** $\ell = \dim \mathbf{T}$) and corresponding **root system** Φ. But within each type there may be several distinct groups. There is always a **simply connected** group $\mathbf{G}$ and an **adjoint** group isomorphic to $\mathbf{G}$ modulo its finite center. There may also be intermediate groups: quotients of $\mathbf{G}$ by central subgroups. The simply connected group is equal to its derived group, while its center is naturally isomorphic to the quotient of the weight lattice X by the root lattice $\mathbb{Z}\Phi$.

1.2. Classification Over Finite Fields

The classification of the groups $\mathbf{G}(k)$ when $\mathbf{G}$ is a simple algebraic group defined over a finite field k begins with the fact (based on Lang's Theorem) that $\mathbf{G}$ is quasisiplit over k. If moreover $\mathbf{G}$ has a k-split maximal torus (isomorphic over k to a product of copies of the multiplicative group), then $\mathbf{G}$ is called **split** over k, otherwise **nonsplit**. Split groups exist for all Lie types, but nonsplit groups only for types A_ℓ (with $\ell > 1$), D_ℓ (with $\ell \geq 4$), and E_6. This classification was worked out independently by Hertzig, Steinberg, and Tits, mainly in the framework of Galois cohomology.

While there are strikingly close parallels in the structure and representations of split and nonsplit groups of the same type over $\mathbb{F}_q$, the split groups are usually much easier to work with. For example, the important subgroups of $\mathbf{G}$ defined over $\mathbb{F}_q$ play a similar role in the structure of $\mathbf{G}(k)$: maximal tori, Borel subgroups, root groups, etc. But for a nonsplit group there are added complications in the description of tori and root groups. It is common in the literature to find separate treatments of split and nonsplit cases.

A further complication is that within most Lie types there are several distinct finite groups, as in the algebraic group classification: for example, $\mathrm{SL}(n,q), \mathrm{PGL}(n,q)$, and $\mathrm{PSL}(n,q)$ are all of type A_{n-1}. When $\mathbf{G}$ is simply connected as in the case of $\mathrm{SL}(n, K)$, one usually calls the finite group $\mathbf{G}(k)$ **universal**. (We remark that Schmid [**358**] proposes a broader concept of "universal" group for arbitrary root systems.)

To show that finite groups of all possible Lie types actually exist, it is efficient to follow Chevalley's uniform method (recalled in 1.5 below). For classical types A–D, one can instead make identifications with known classical matrix groups over k. A careful review of the concrete descriptions of classical groups over finite fields can be found in the Atlas [**117**, Chap. 2].

The reader should be cautioned that varying notational schemes are found in the literature. For example, the notation $\mathrm{SU}(n, q)$ is used here for the subgroup of $\mathrm{GL}(n, q^2)$ which is often denoted more literally as $\mathrm{SU}(n, q^2)$. Our choice emphasizes the close parallel between $\mathrm{SL}(n, q)$ and $\mathrm{SU}(n, q)$, seen in their order formulas and representation theories. More functorial notation such as $\mathrm{SL}_n(\mathbb{F}_q)$ or $\mathrm{SL}_n(q)$ is also widely used.

In dealing with modular representations in the defining characteristic, it is convenient to work in the setting of simply connected simple algebraic groups. Here the main unsolved problems about characters and dimensions can be formulated efficiently in the language of weights. To make the transition to closely related groups such as quotients by central subgroups and their derived groups, one has to use standard representation-theoretic techniques unrelated to Lie theory. Usually this is routine for the problems we study here. But in some situations the transition can be delicate, as seen for example in the work of Lusztig and others on ordinary characters. (It is instructive to study the organization of character tables for families of related groups in the Atlas [**117**].)

Sometimes it is more natural to study arbitrary semisimple groups, or reductive groups such as $\mathrm{GL}(n, K)$ which are not semisimple. Though we typically formulate results only for simple algebraic groups, the reader should be aware of a few complications. For example, the algebraic group $\mathrm{PGL}(n, K) = \mathrm{GL}(n, K)/K^\times$ is isomorphic to the simple adjoint group of type A_{n-1}, which over an algebraically closed field is the same as $\mathrm{PSL}(n, K)$. However, over a finite subfield there is usually a difference between the group $\mathrm{PGL}(n, k)$ and its subgroup $\mathrm{PSL}(n, k)$, the latter typically being the derived group of the former and having index equal to the order of the group of nth roots of 1 in $k^\times$.

1.3. Frobenius Maps

The algebraic group approach does not directly yield the groups of Suzuki and Ree. To unify the description of all finite groups of Lie type, Steinberg [**395**] studied an arbitrary algebraic group endomorphism $\sigma : \mathbf{G} \to \mathbf{G}$ whose group of fixed points $\mathbf{G}_\sigma$ is finite. Here $\mathbf{G}$ is defined and split over $\mathbb{F}_q$. The most basic example is the **standard Frobenius map** relative to $q = p^r$: if $\mathbf{G}$ is given explicitly as a matrix group, this map just raises each matrix entry to the qth power. (The map can be described intrinsically in terms of the algebra of regular functions on $\mathbf{G}$.) The resulting finite group of fixed points coincides with the group of rational points $\mathbf{G}(\mathbb{F}_q)$.

More complicated endomorphisms are obtained by composing the standard Frobenius map relative to q with a nontrivial graph automorphism π arising from a symmetry of the Dynkin diagram of $\mathbf{G}$. (These maps commute.) The only simple groups with a nontrivial graph automorphism are those of types A_ℓ (with $\ell > 1$),

D_ℓ (for $\ell \geq 4$), and E_6. Type D_4 is unusual in having such automorphisms of both order 2 and order 3. The group of fixed points is isomorphic to the group of rational points over $\mathbb{F}_q$ of a quasisplit but nonsplit group of the same type as $\mathbf{G}$. For example, one gets $\mathrm{SU}(3,q)$ rather than $\mathrm{SL}(3,q)$ from the ambient algebraic group $\mathrm{SL}(3,K)$. Note that the square or cube of the endomorphism is the standard Frobenius map relative to q.

In each of these cases the endomorphism $\mathbf{G} \to \mathbf{G}$ characterizes the $\mathbb{F}_q$-structure of $\mathbf{G}$. Following current usage, we denote the map by F and call it a **Frobenius map** relative to p^r (and π if a nontrivial graph automorphism is involved). Write $G := \mathbf{G}^F$ for the finite group of fixed points of F.

For groups with root systems of types $\mathrm{B}_2, \mathrm{F}_4, \mathrm{G}_2$, more exotic endomorphisms of $\mathbf{G}$ can be constructed, yielding the groups of Suzuki in type B_2 and Ree in types F_4 and G_2. (See 5.3 and Chapter 20 for further details.) One combines selective qth powers with graph symmetries interchanging long and short root subgroups. More precisely, let $q = p^{2r+1}$ be an odd power of $p = 2$ for types B_2 and F_4 (resp. $p = 3$ for type G_2), and assume $\mathbf{G}$ is defined over $\mathbb{F}_q$ (hence split). Then let $\alpha \mapsto \overline{\alpha}$ interchange short and long roots as a graph automorphism would do with lengths ignored. Map an element $x_\alpha(c)$ of a root subgroup $\mathbf{U}_\alpha$ to $x_{\overline{\alpha}}(c')$, where $c' := c^{p^r}$ if α is long and $c' := c^{p^{r+1}}$ if α is short. For the chosen p this defines an endomorphism of $\mathbf{G}$ whose square is just the standard Frobenius map corresponding to q. (Some authors call this type of endomorphism a "Frobenius map", but we do not.)

The upshot of Steinberg's analysis is that these are the only possible endomorphisms of $\mathbf{G}$ having a finite fixed point subgroup. The language of Frobenius maps is used systematically (but with some variation in the definition) in most current work. See for example Cabanes–Enguehard [**78**], Carter [**86**, 14.1], Digne–Michel [**134**, Chap. 3], Geck [**184**], Gorenstein–Lyons–Solomon [**190**, 2.1].

1.4. Lang Maps

The basic theorem of Lang mentioned earlier can be reformulated for a connected algebraic group $\mathbf{G}$ defined over $\mathbb{F}_q$ in the more general form suggested by Steinberg [**395**, §10]. Starting with a Frobenius map $F : \mathbf{G} \to \mathbf{G}$ relative to q, the associated **Lang map** $L : \mathbf{G} \to \mathbf{G}$ is defined by $L(g) = F(g)g^{-1}$. Then the theorem states that L is *surjective*. Here F may be standard or may (in case $\mathbf{G}$ is reductive) involve also a graph automorphism.

In the literature there are several variants of the definition of L using different left–right conventions, for example, $g \mapsto g^{-1}F(g)$ or $g \mapsto gF(g^{-1})$. This has no effect on the surjectivity or its applications.

The proof of Lang's Theorem uses the fact that the fibers of L are the orbits of $G = \mathbf{G}^F$ acting by right translation, together with an easy calculation showing that the differential at each point is bijective. See for example Springer [**385**, 4.4.17]. The argument translates also into the language of quotients: The quotient of the affine variety $\mathbf{G}$ by the right translation action of G is isomorphic to $\mathbf{G}$ itself, with L as the quotient map. In particular, the G-invariant regular functions on $\mathbf{G}$ are of the form $L^*f = f \circ L$ for $f \in K[\mathbf{G}]$. (These statements apply equally well to the restricted Lang map on a closed connected subgroup $\mathbf{H}$ of $\mathbf{G}$.)

1.5. Chevalley Groups and Twisted Groups

To construct all possible finite groups of Lie type in a uniform way, one follows the lead of Chevalley's 1955 Tôhoku paper. He showed how to obtain the simple groups of split type by a process of reduction modulo p. This starts with the choice of a good $\mathbb{Z}$-basis for any simple Lie algebra over $\mathbb{C}$, whose adjoint operators raised to powers and divided by corresponding factorials still leave the basis invariant. The operators corresponding to root vectors act nilpotently, so the usual exponential power series is just a polynomial operator leaving the Chevalley basis invariant. It makes sense to reduce modulo a prime p (no division by p being required), leading to a matrix group over $\mathbb{F}_p$ generated by "exponentials". Extension of scalars gives the desired groups over arbitrary finite fields.

With a few very small exceptions, these groups are simple and are the derived groups of the various $\mathbf{G}(\mathbb{F}_q)$ obtained from split adjoint groups (for example $\mathrm{PSL}(n, q) \leq \mathrm{PGL}(n, q)$). The exceptions are the solvable groups $\mathrm{A}_1(2)$ and $\mathrm{A}_1(3)$; the group $\mathrm{B}_2(2)$ of order 720, which is isomorphic to the symmetric group S_6; and the group $\mathrm{G}_2(2)$ of order 12096, which has a simple subgroup of index 2 isomorphic to ${}^2\mathrm{A}_2(3)$.

In his 1967–68 Yale lectures [**394**], Steinberg replaced the adjoint representation by an arbitrary irreducible representation over $\mathbb{C}$ and generalized Chevalley's procedure to obtain groups of all isogeny types as well as the various kinds of twisted groups. (See also Carter's book [**86**].)

Unless other specified, we always work with a simply connected group $\mathbf{G}$, defined and split over $\mathbb{F}_p$. The finite group $G = \mathbf{G}^F$ of fixed points under the standard Frobenius map relative to q is called a (universal) **Chevalley group**, while the term **twisted group** refers to the fixed points under a Frobenius map involving a nontrivial graph automorphism. But note that some authors include all of these groups under the rubric "Chevalley group". And it is common to regard the groups of Suzuki and Ree as types of twisted groups, but we find it more convenient to keep these separate.

Table 1 summarizes our labelling by Lie type, together with the order of the corresponding universal group. Chevalley groups are listed first, followed by twisted groups and the groups of Suzuki and Ree.

As in the case of classical groups, notational conventions for Lie types vary in the literature: see for example Gorenstein–Lyons–Solomon [**188**, pp. 8–9] and the Atlas of Finite Groups [**117**]. In the case of the twisted groups of types $\mathrm{A}, \mathrm{D}, \mathrm{E}_6$, some sources use q^2 rather than q in the labels. And in the case of the groups of Suzuki and Ree, q^2 may be replaced by q in the labels as well as the order formulas. This issue is discussed carefully in the Atlas [**117**, 3.2], where alternate notations (such as $\mathrm{Sz}(q)$ for ${}^2\mathrm{B}_2(q^2)$) are also described. We prefer notation which shows the analogy between orders of the twisted and nontwisted groups.

1.6. Example: SL(3, q) and SU(3, q)

The similarities and differences between split and nonsplit groups of the same type show up clearly in the simplest situation, groups of types $\mathrm{A}_2(q)$ and ${}^2\mathrm{A}_2(q)$. With our notational conventions, the orders of the two groups are given by similar polynomials of degree $8 = \dim \mathrm{SL}(3, K)$:

$$|\,\mathrm{SL}(3, q)| = q^3(q^2 - 1)(q^3 - 1) \text{ and } |\,\mathrm{SU}(3, q)| = q^3(q^2 + 1)(q^3 - 1).$$

Lie type	Order of universal group
$\mathrm{A}_\ell(q), \ell \geq 1$	$q^{\binom{\ell+1}{2}} \prod_{i=2}^{\ell+1}(q^i - 1)$
$\mathrm{B}_\ell(q), \ell \geq 2$	$q^{\ell^2} \prod_{i=1}^{\ell}(q^{2i} - 1)$
$\mathrm{C}_\ell(q), \ell \geq 2$	$q^{\ell^2} \prod_{i=1}^{\ell}(q^{2i} - 1)$
$\mathrm{D}_\ell(q), \ell \geq 4$	$q^{\ell(\ell-1)}(q^\ell - 1) \prod_{i=1}^{\ell-1}(q^{2i} - 1)$
$\mathrm{E}_6(q)$	$q^{36}(q^2-1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)$
$\mathrm{E}_7(q)$	$q^{63}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)$
$\mathrm{E}_8(q)$	$q^{120}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)$
$\mathrm{F}_4(q)$	$q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$
$\mathrm{G}_2(q)$	$q^6(q^2-1)(q^6-1)$
${}^2\mathrm{A}_\ell(q), \ell \geq 2$	$q^{\binom{\ell+1}{2}} \prod_{i=2}^{\ell+1}(q^i - (-1)^i)$
${}^2\mathrm{D}_\ell(q), \ell \geq 4$	$q^{\ell(\ell-1)}(q^\ell + 1) \prod_{i=1}^{\ell-1}(q^{2i} - 1)$
${}^3\mathrm{D}_4(q)$	$q^{12}(q^2-1)(q^6-1)(q^8+q^4+1)$
${}^2\mathrm{E}_6(q)$	$q^{36}(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1)$
${}^2\mathrm{B}_2(q^2), q^2 = 2^{2r+1}$	$q^4(q^2-1)(q^4+1)$
${}^2\mathrm{F}_4(q^2), q^2 = 2^{2r+1}$	$q^{24}(q^2-1)(q^6+1)(q^8-1)(q^{12}+1)$
${}^2\mathrm{G}_2(q^2), q^2 = 3^{2r+1}$	$q^6(q^2-1)(q^6+1)$

TABLE 1. Universal groups of Lie type

Consider how the finite torus $\mathbf{T}^F$ looks in each case. In $\mathrm{SL}(3,q)$ we just get a direct product of two copies of $\mathbb{F}_q^\times$, one for each simple root α, β. But in $\mathrm{SU}(3,q) \subset \mathrm{SL}(3,q^2)$ the situation is quite different. Here the graph automorphism interchanges α and β, so a typical diagonal matrix fixed by F has eigenvalues $c, c^q, 1/c^{q+1}$ as c runs over the cyclic group $\mathbb{F}_{q^2}^\times$ of order q^2-1. Thus $\mathbf{T}^F$ is cyclic.

There is also a sharp contrast in the structure of the upper triangular unipotent subgroup $\mathbf{U}^F$ in the two cases. This subgroup of $\mathrm{SL}(3,q)$ (a Sylow p-subgroup) has order q^3, with structure like that of $\mathbf{U}$ as a product of three root groups. In $\mathrm{SU}(3,q)$ the order of $\mathbf{U}^F$ is the same, but the group structure changes: a unipotent matrix

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with entries in $\mathbb{F}_{q^2}$ is fixed by F precisely when $b = a^q$ and $c + c^q + a^{q+1} = 0$. Here there is a single positive "root group".

Differences like these propagate through all pairs of split and nonsplit groups based on the same root system. See Carter [**86**, Chap. 13] or Steinberg [**394**, §11] for full details about the structure of twisted groups (along with those of Suzuki and Ree).

1.7. Groups With a BN-Pair

In work leading to the classification of finite simple groups, it has often been useful to describe groups of Lie type in a uniform axiomatic setting independent of algebraic groups: see Gorenstein–Lyons–Solomon [**189**, §30]. For this purpose the most efficient formalism is that of **BN-pairs** (called **Tits systems** by Bourbaki in [**56**, Chap. IV]), as introduced by Tits in the aftermath of Chevalley's 1955 Tôhoku paper. This captures the essence of the Bruhat decomposition as it appears in various settings, without explicit introduction of the root system.

Given a group G with subgroups B and N, the data defining a BN-pair consists formally of a quadruple (G, B, N, S) subject to the following requirements:

- G is generated by its subgroups B and N.
- $T := B \cap N$ is a normal subgroup of N.
- $W := N/T$ is generated by a set S of involutions.
- For $s \in S$ and $w \in W$, $sBw \subseteq BwB \cup BswB$.
- For $s \in S$, $sBs \neq B$.

Here we adopt the usual convention of writing expressions like sBw and BwB when the choice of a representative in N of an element of W makes no difference. (Note that the letter H is often used in the literature for the group we call T.)

The axioms lead quickly to the **Bruhat decomposition**: G is the disjoint union (indexed by W) of the double cosets BwB. As a further consequence of the axioms, W is seen to be a **Coxeter group** with distinguished set of generators S. The cardinality of S is then called the **rank** of the BN-pair.

A minor adjustment can be made without significant loss of generality, to insure that the BN-pair is "saturated": $T = \bigcap_{w \in W} wBw^{-1}$. By the axioms, T is always included in the right side. If the inclusion were proper, we could simply replace T by the right side and enlarge N accordingly. (We always assume saturation.)

With our previous notational conventions, the group $G = \mathbf{G}^F$ has a natural BN-pair structure: When G is a Chevalley group, take $B = \mathbf{B}^F$ for a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ corresponding to Φ^+ and $N = \mathbf{N}^F$ for $\mathbf{N} = N_{\mathbf{G}}(\mathbf{T})$, the normalizer of an $\mathbb{F}_q$-split maximal torus $\mathbf{T}$ lying in $\mathbf{B}$. But when G is a twisted group, the BN-pair structure encodes the "relative" structure of a nonsplit algebraic group over $\mathbb{F}_q$: the Weyl group of the BN-pair is the subgroup of the usual Weyl group fixed by the induced action of F, while the rank of the BN-pair is the "relative" rank of the algebraic group measuring the dimension of a maximal $\mathbb{F}_q$-split torus. For example, one views $\mathrm{SU}(3,q)$ as having a BN-pair of rank 1.

Which finite groups—especially simple groups—have a BN-pair? A group with a BN-pair of rank 1 is the same thing as a doubly transitive permutation group: B can be interpreted as the isotropy group of a point, while N stabilizes a set consisting of this point and one other (and T is the subgroup fixing both). Thus it is unrealistic to expect a complete list of finite groups with such a BN-pair. But surprisingly enough, Tits [**420**] was able to show (by geometric methods) that all finite simple groups with a BN-pair of rank ≥ 3 are of Lie type.

A more serious restriction is needed in rank 2 to reflect the special internal structure of a group of Lie type: one needs a (saturated) **split BN-pair of characteristic p**. For this, add to the axioms the assumption that B has a normal p-subgroup U complementary to T, while T is abelian and has order relatively prime to p. Using an impressive array of group-theoretic methods, it has been shown that all simple groups with a split BN-pair of characteristic p are of Lie type (or degenerate versions thereof): see Hering–Kantor–Seitz [**204**], Kantor–Seitz [**266**], Fong–Seitz [**175**].

In order to bypass this rather sophisticated literature, one can impose the further condition that the BN-pair is **strongly split** as defined by Cabanes–Enguehard [**78**, §2] and Genet [**185**] (see 8.7 below). This yields Levi decompositions for parabolic subgroups as well as a version of the Chevalley commutation formulas. (See also Tinberg [**418**].)

There are numerous treatments in the literature of groups with a BN-pair, directed toward different applications. See for example Bourbaki [**56**, Chap. IV], Cabanes–Enguehard [**78**, Part I], Carter [**86**, 8.2–8.3] and [**89**, Chap. 2], Curtis [**121, 123**] and [**122**, §3], Curtis–Reiner [**127**, §65, §69], Gorenstein–Lyons–Solomon [**189**, §30], Humphreys [**220**, §29].

1.8. Notational Conventions

Unless otherwise specified, $\mathbf{G}$ will denote a simple, simply connected algebraic group defined and split over $\mathbb{F}_p$, identified with its group $\mathbf{G}(K)$ of rational points over an algebraically closed field K of characteristic $p > 0$. Its root system is Φ, with a simple system Δ. Usually we fix a Borel subgroup $\mathbf{B}$ (corresponding to a set of positive roots Φ^+) and a maximal torus $\mathbf{T} \subset \mathbf{B}$, both split over $\mathbb{F}_p$. The rank of $\mathbf{G}$ is $\ell = \dim \mathbf{T} = |\Delta|$.

Thus $\mathbf{G} \cong \mathrm{SL}(\ell+1, K)$, $\mathrm{Spin}(2\ell+1, K)$, $\mathrm{Sp}(2\ell, K)$, $\mathrm{Spin}(2\ell, K)$, or one of the five exceptional groups of types $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$.

If $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius map relative to q (and possibly a nontrivial graph automorphism π), its fixed point subgroup $G := \mathbf{G}^F$ is a universal Chevalley group or a twisted group of type A_ℓ $(\ell > 1), \mathrm{D}_\ell$, or E_6 over $\mathbb{F}_q$. (Suzuki and Ree groups are discussed separately in Chapter 20.)

Additional notation will be introduced as we go along. (See the list of frequently used notation at the end of the book.)

CHAPTER 2

Simple Modules

To study the representations over K of an arbitrary finite group G, one usually concentrates first on those which are realized within the group algebra KG. The main examples are simple modules and projective modules.

When G is a finite group of Lie type, there are two natural approaches to the study of simple KG-modules:

- Describe them intrinsically in the setting of groups with split BN-pairs.
- Describe them as restrictions of simple modules for the ambient algebraic group $\mathbf{G}$.

While the second approach is less direct, it has yielded (so far) much more detailed information than the first approach and will therefore be our main focus here. We defer until Chapter 7 the more self-contained development due to Curtis and Richen, based on BN-pairs.

Even though it is possible to classify the simple KG-modules in a coherent way from the algebraic group viewpoint, we still do not know in most cases their dimensions or (Brauer) characters. Modulo knowledge of the formal characters of simple $\mathbf{G}$-modules (still incomplete in most cases), which we call **standard character data** for $\mathbf{G}$, it is often possible to derive further results about the category of finite dimensional KG-modules: projectives, extensions, etc. This is usually the approach we follow, motivated by Lusztig's Conjecture for $\mathbf{G}$ (see 3.11 below).

After a detailed review of simple modules for the algebraic groups, following [**RAGS**], we turn to the finite groups. Besides the cited papers, we can point to useful surveys in the Atlas [**117**, Chap. 2] and Gorenstein–Lyons–Solomon [**190**, 2.8]. For unexplained notation see 1.8.

2.1. Representations and Formal Characters

For the study of finite groups of Lie type we normally find it most convenient to work with a simple algebraic group. But the treatment of representations in [**RAGS**] allows $\mathbf{G}$ to be semisimple—or even reductive, which is needed for inductive purposes when passing to Levi subgroups.

In this and the following section, $\mathbf{G}$ can be any connected semisimple group over K, with a fixed Borel subgroup $\mathbf{B} = \mathbf{TU}$ corresponding to a choice of simple system Δ and positive roots Φ^+. For brevity we always denote by X the character group $X(\mathbf{T})$ (with the convention that the group law is written additively) and by X^+ the subset of dominant weights relative to Φ^+. Then X is partially ordered by $\mu \leq \lambda$ iff $\lambda - \mu$ is a sum (possibly 0) of positive roots. We call this the **natural ordering** of X.

Denote by W the **Weyl group** $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ of $\mathbf{G}$ relative to the maximal torus $\mathbf{T}$. So W is generated by "simple" reflections (relative to roots in Δ) and has a corresponding length function $\ell(w)$. There is a unique longest element $w_\circ$ in W, which sends Φ^+ to $-\Phi^+$; its length is $m := |\Phi^+|$.

The dual root system is denoted by $\Phi^\vee$. When Φ is irreducible, $\Phi^\vee$ has the same type as Φ except that types B_ℓ and C_ℓ are dual to each other. The natural pairing $\langle x, \alpha^\vee \rangle$ is defined for all $x \in \mathbb{R} \otimes_{\mathbb{Z}} X$.

Representations of $\mathbf{G}$ are always assumed to be *rational* and *finite dimensional*, unless otherwise specified. Typically we use the equivalent language of $\mathbf{G}$-modules. It is a basic fact that Jordan decomposition in $\mathbf{G}$ is preserved under homomorphisms of algebraic groups. This implies that any $\mathbf{G}$-module M is a direct sum of its **weight spaces** M_λ ($\lambda \in X$) relative to $\mathbf{T}$:

$$M_\lambda := \{x \in M \mid t \cdot x = \lambda(t)x \text{ for all } t \in \mathbf{T}\}.$$

In turn, M has a **formal character** $\operatorname{ch} M$ in the group ring $\mathbb{Z}[X]$ of X. To view X as a multiplicative group in this context, we introduce canonical basis elements $e(\lambda)$ of the free abelian group $\mathbb{Z}[X]$ indexed by $\lambda \in X$. These are multiplied by the rule $e(\lambda)e(\mu) = e(\lambda + \mu)$. Now the formal character is defined by

$$\operatorname{ch} M := \sum_{\lambda \in X} \dim M_\lambda \, e(\lambda).$$

Finite dimensionality of M implies that $\dim M_\lambda = \dim M_{w\lambda}$ for all $w \in W$, so $\operatorname{ch} M$ lies in the subring of W-invariants $\mathcal{X} := \mathbb{Z}[X]^W$. We call this the **formal character ring** of $\mathbf{G}$ relative to $\mathbf{T}$. As a result, $\operatorname{ch} M$ can be rewritten as a $\mathbb{Z}^+$-linear combination of various orbit sums $s(\mu) := \sum_{w \in W^\mu} e(w\mu)$, with $\mu \in X^+$ and W^μ a set of coset representatives of W modulo the isotropy group W_μ of μ.

For the study of representations it is usually most convenient to assume that $\mathbf{G}$ is *simply connected*. This translates into the assumption that X is the full weight lattice of the abstract root system Φ. Other semisimple groups with the same root system are obtained by factoring out subgroups of the finite group $Z(\mathbf{G})$. Then it is not difficult to sort out which representations of $\mathbf{G}$ induce representations of the quotient group: it is just a question of which weights lie in the character group of the corresponding quotient of $\mathbf{T}$ (a sublattice of X).

2.2. Simple Modules for Algebraic Groups

As shown by Chevalley in the late 1950s, the highest weight classification of irreducible representations for a semisimple algebraic group over an algebraically closed field is essentially characteristic-free. (See [**RAGS**, II.2], [**220**, §31].)

THEOREM. *Let* $\mathbf{G}$ *be a semisimple algebraic group over* K. *Fix notation as in* 2.1, *and let* $\mathbf{B}^- = \mathbf{T}\mathbf{U}^-$ *be the Borel subgroup of* $\mathbf{G}$ *opposite to* $\mathbf{B}$. *Then:*

(a) *Every simple (rational)* $\mathbf{G}$*-module* M *has a unique highest weight* $\lambda \in X^+$ *in the natural partial ordering of* X. *Whenever* $M_\mu \neq 0$, *we have* $\mu \leq \lambda$.
(b) *The weight space* M_λ *is one-dimensional, spanned by a* $\mathbf{U}$*-invariant vector* v^+, *called a* ***maximal vector*** *or* ***highest weight vector***.
(c) M *is spanned by the vectors* $u \cdot v^+$, *for* $u \in \mathbf{U}^-$.
(d) *Two simple modules with the same highest weight* λ *are isomorphic, so* M *may be denoted unambiguously by* $L(\lambda)$.
(e) *For every* $\lambda \in X^+$ *there exists a simple* $\mathbf{G}$*-module of highest weight* λ.