

Cambridge University Press  
978-0-521-67109-5 - Applied Solid Mechanics  
Peter Howell, Gregory Kozyreff and John Ockendon  
Frontmatter  
[More information](#)

---

## APPLIED SOLID MECHANICS

Much of the world around us, both natural and man-made, is built from and held together by solid materials. Understanding how they behave is the task of solid mechanics, which can in turn be applied to a wide range of areas from earthquake mechanics and the construction industry to biomechanics. The variety of materials (such as metals, rocks, glasses, sand, flesh and bone) and their properties (such as porosity, viscosity, elasticity, plasticity) are reflected by the concepts and techniques needed to understand them, which are a rich mixture of mathematics, physics, experiment and intuition. These are all brought to bear in this distinctive book, which is based on years of experience in research and teaching. Theory is related to practical applications, where surprising phenomena occur and where innovative mathematical methods are needed to understand features such as fracture. Starting from the very simplest situations, based on elementary observations in engineering and physics, models of increasing sophistication are derived and applied. The emphasis is on problem solving and on building an intuitive understanding, rather than on a technical presentation of theoretical topics. The text is complemented by over 100 carefully chosen exercises, and the minimal prerequisites make it an ideal companion for mathematics students taking advanced courses, for those undertaking research in the area or for those working in other disciplines in which solid mechanics plays a crucial role.

Cambridge University Press  
978-0-521-67109-5 - Applied Solid Mechanics  
Peter Howell, Gregory Kozyreff and John Ockendon  
Frontmatter  
[More information](#)

---

## Cambridge Texts in Applied Mathematics

### **Editorial Board**

Mark Ablowitz, *University of Colorado, Boulder*

S. Davis, *Northwestern University*

E. J. Hinch, *University of Cambridge*

Arieh Iserles, *University of Cambridge*

John Ockendon, *University of Oxford*

Peter Olver, *University of Minnesota*

Cambridge University Press  
978-0-521-67109-5 - Applied Solid Mechanics  
Peter Howell, Gregory Kozyreff and John Ockendon  
Frontmatter  
[More information](#)

---

# APPLIED SOLID MECHANICS

PETER HOWELL

*University of Oxford*

GREGORY KOZYREFF

*Fonds de la Recherche Scientifique—FNRS  
and Université Libre de Bruxelles*

JOHN OCKENDON

*University of Oxford*



**CAMBRIDGE**  
UNIVERSITY PRESS

Cambridge University Press  
 978-0-521-67109-5 - Applied Solid Mechanics  
 Peter Howell, Gregory Kozyreff and John Ockendon  
 Frontmatter  
[More information](#)

CAMBRIDGE UNIVERSITY PRESS  
 Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi  
 Cambridge University Press  
 The Edinburgh Building, Cambridge CB2 8RU, UK  
 Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)  
 Information on this title: [www.cambridge.org/9780521854894](http://www.cambridge.org/9780521854894)

© P. D. Howell, G. Kozyreff and J. R. Ockendon 2009

This publication is in copyright. Subject to statutory exception  
 and to the provisions of relevant collective licensing agreements,  
 no reproduction of any part may take place without  
 the written permission of Cambridge University Press.

First published 2009

Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this publication is available from the British Library*

*Library of Congress Cataloguing in Publication data*

Howell, Peter, 1970–  
 Applied solid mechanics / Peter Howell, John Ockendon, Gregory Kozyreff.  
 p. cm.

Includes bibliographical references and index.

ISBN 978-0-521-85489-4 (hardback : alk. paper) – ISBN 978-0-521-67109-5 (paperback : alk. paper)  
 1. Materials – Mechanical properties. 2. Deformations (Mechanics) 3. Elastic solids. I. Ockendon, J. R.  
 II. Kozyreff, Gregory. III. Title.  
 TA404.8.H69 2009  
 620.1'05 – dc22 2008032880

ISBN 978-0-521-85489-4 hardback  
 ISBN 978-0-521-67109-5 paperback

---

Cambridge University Press has no responsibility for the persistence or  
 accuracy of URLs for external or third-party internet websites referred to  
 in this publication, and does not guarantee that any content on such  
 websites is, or will remain, accurate or appropriate.

---

## Contents

<i>List of illustrations</i>	<i>page</i> viii
<i>Prologue</i>	xiii
<b>Modelling solids</b>	1
1.1 Introduction	1
1.2 Hooke's law	2
1.3 Lagrangian and Eulerian coordinates	3
1.4 Strain	4
1.5 Stress	7
1.6 Conservation of momentum	10
1.7 Linear elasticity	11
1.8 The incompressibility approximation	13
1.9 Energy	14
1.10 Boundary conditions and well-posedness	16
1.11 Coordinate systems	19
Exercises	24
<b>Linear elastostatics</b>	28
2.1 Introduction	28
2.2 Linear displacements	29
2.3 Antiplane strain	37
2.4 Torsion	39
2.5 Multiply-connected domains	42
2.6 Plane strain	47
2.7 Compatibility	68
2.8 Generalised stress functions	70
2.9 Singular solutions in elastostatics	82
2.10 Concluding remark	93
Exercises	93

<b>Linear elastodynamics</b>	103
3.1 Introduction	103
3.2 Normal modes and plane waves	104
3.3 Dynamic stress functions	121
3.4 Waves in cylinders and spheres	124
3.5 Initial-value problems	132
3.6 Moving singularities	138
3.7 Concluding remarks	143
Exercises	143
<b>Approximate theories</b>	150
4.1 Introduction	150
4.2 Longitudinal displacement of a bar	151
4.3 Transverse displacements of a string	152
4.4 Transverse displacements of a beam	153
4.5 Linear rod theory	158
4.6 Linear plate theory	162
4.7 Von Kármán plate theory	172
4.8 Weakly curved shell theory	177
4.9 Nonlinear beam theory	187
4.10 Nonlinear rod theory	195
4.11 Geometrically nonlinear wave propagation	198
4.12 Concluding remarks	204
Exercises	205
<b>Nonlinear elasticity</b>	215
5.1 Introduction	215
5.2 Stress and strain revisited	216
5.3 The constitutive relation	221
5.4 Examples	233
5.5 Concluding remarks	239
Exercises	239
<b>Asymptotic analysis</b>	245
6.1 Introduction	245
6.2 Antiplane strain in a thin plate	246
6.3 The linear plate equation	248
6.4 Boundary conditions and Saint-Venant's principle	253
6.5 The von Kármán plate equations	261
6.6 The Euler–Bernoulli plate equations	267
6.7 The linear rod equations	273
6.8 Linear shell theory	278

<i>Contents</i>		vii
6.9	Concluding remarks	282
	Exercises	283
	<b>Fracture and contact</b>	287
7.1	Introduction	287
7.2	Static brittle fracture	288
7.3	Contact	309
7.4	Concluding remarks	320
	Exercises	321
	<b>Plasticity</b>	328
8.1	Introduction	328
8.2	Models for granular material	330
8.3	Dislocation theory	337
8.4	Perfect plasticity theory for metals	344
8.5	Kinematics	358
8.6	Conservation of momentum	360
8.7	Conservation of energy	360
8.8	The flow rule	362
8.9	Simultaneous elasticity and plasticity	364
8.10	Examples	365
8.11	Concluding remarks	370
	Exercises	372
	<b>More general theories</b>	378
9.1	Introduction	378
9.2	Viscoelasticity	379
9.3	Thermoelasticity	388
9.4	Composite materials and homogenisation	391
9.5	Poroelasticity	408
9.6	Anisotropy	413
9.7	Concluding remarks	417
	Exercises	417
	<i>Epilogue</i>	426
<i>Appendix</i>	<b>Orthogonal curvilinear coordinates</b>	428
	<i>References</i>	440
	<i>Index</i>	442

## Illustrations

1.1	A reference tetrahedron.	<i>page</i> 8
1.2	The forces acting on a small two-dimensional element.	9
1.3	A small pill-box-shaped region at the boundary between two elastic solids.	18
1.4	Forces acting on a polar element of solid.	22
1.5	A system of masses connected by springs.	25
2.1	A unit cube undergoing (a) uniform expansion, (b) one-dimensional shear, (c) uniaxial stretching.	30
2.2	A uniform bar being stretched under a tensile force.	32
2.3	A paper model with negative Poisson's ratio.	33
2.4	A strained plate.	34
2.5	A bar in a state of antiplane strain.	38
2.6	A twisted bar.	39
2.7	A uniform tubular torsion bar.	43
2.8	The cross-section of (a) a circular cylindrical tube; (b) a cut tube.	44
2.9	The unit normal and tangent to the boundary of a plane region.	49
2.10	A plane annulus being inflated by an internal pressure.	53
2.11	A plane rectangular region subject to tangential tractions on its faces.	57
2.12	The tractions applied to the edge of a semi-infinite strip.	59
2.13	The surface displacement of a half-space and corresponding surface pressure.	65
2.14	A family of functions $\delta_\varepsilon(x)$ that approach a delta-function as $\varepsilon \rightarrow 0$ .	83
2.15	Contours of the maximum shear stress created by a point force acting at the origin.	85
2.16	Four point forces.	91
3.1	Plots of the first three Bessel functions.	108
3.2	A $P$ -wave reflecting from a rigid boundary.	116
3.3	A layered elastic medium.	117
3.4	Dispersion relation for symmetric and antisymmetric Love waves.	120
3.5	Illustration of flexural waves.	128
3.6	The one-dimensional fundamental solution.	134
3.7	The two-dimensional fundamental solution.	135



*List of illustrations*

ix

3.8	The cone $x^2 + y^2 = c^2t^2$ tangent to the plane $k_1x + k_2y = \omega t$ .	138
3.9	The two-sheeted characteristic cone for the Navier equation.	138
3.10	The response of a string to a point force moving at speed $V$ .	139
3.11	Wave-fronts generated by a moving force on an elastic membrane.	141
3.12	$P$ -wave- and $S$ -wave-fronts generated by a point force moving at speed $V$ in plane strain.	142
3.13	Group velocity versus wave-number for symmetric and antisymmetric Love waves.	146
4.1	The forces acting on a small length of a uniform bar.	151
4.2	The forces acting on a small length of an elastic string.	153
4.3	The forces and moments acting on a small segment of an elastic beam.	154
4.4	The end of a beam under clamped, simply supported and free conditions.	155
4.5	The first three buckling modes of a clamped elastic beam.	157
4.6	The internal force components in a thin elastic rod.	159
4.7	Cross-section through a rod showing the bending moment components.	159
4.8	Examples of cross-sections in the $(y, z)$ -plane and their bending stiffnesses.	161
4.9	The forces acting on a small section of an elastic plate.	163
4.10	The bending moments acting on a section of an elastic plate.	164
4.11	The displacement of a simply supported rectangular plate sagging under gravity.	169
4.12	(a) A cylinder, (b) a cone, (c) another developable surface, (d) a hyperboloid.	175
4.13	Typical surface shapes with (a) zero, (b) negative and (c) positive Gaussian curvature.	179
4.14	Deformations of a cylindrical shell.	184
4.15	Deformations of an anticlastic shell.	185
4.16	Deformations of a synclastic shell.	186
4.17	A beam (a) before and (b) after bending; (c) a close-up of the displacement field.	187
4.18	(a) The forces and moments acting on a small segment of a beam. (b) The sign convention for the forces at the ends of the beam.	188
4.19	(a) Final angle of a diving board versus applied force parameter. (b) Deflection of a diving board for various values of the force parameter.	191
4.20	(a) Response diagram of the amplitude of the linearised solution for a buckling beam versus the force parameter. (b) Corresponding response of the weakly nonlinear solution.	193
4.21	(a) Pitchfork bifurcation diagram of leading-order amplitude versus forcing parameter. (b) The corresponding diagram when asymmetry is introduced.	195
4.22	A system of pendulums attached to a twisting rubber band.	199
4.23	A kink propagating along a series of pendulums attached to a rod.	200
4.24	Travelling wave solution of the nonlinear beam equations.	201
4.25	A beam clamped near the edge of a table.	206
4.26	A beam supported at two points.	207
4.27	The first three buckling modes of a vertically clamped beam.	213

x	<i>List of illustrations</i>	
5.1	The deformation of a small scalene cylinder.	218
5.2	Typical force–strain graphs for uniaxial tests on various materials.	233
5.3	A square membrane subject to an isotropic tensile force.	234
5.4	Response diagrams for a biaxially-loaded incompressible sheet of Mooney–Rivlin material.	235
5.5	Scaled pressure inside a balloon as a function of the stretch for various values of the Mooney–Rivlin parameter.	236
5.6	Gas pressure inside a cavity as a function of inflation coefficient for various values of the Mooney–Rivlin parameter.	238
6.1	The edge of a plate subject to tractions.	254
6.2	The geometry of a deformed two-dimensional plate.	269
7.1	Definition sketch of a thin crack.	288
7.2	Definition sketch for contact between two solids.	288
7.3	(a) A Mode III crack. (b) A cross-section in the $(x, y)$ -plane.	290
7.4	Definition sketch for the function $\sqrt{z^2 - c^2}$ .	292
7.5	Displacement field for a Mode III crack.	293
7.6	(a) A planar Mode II crack. (b) The regularised problem of a thin elliptical crack.	297
7.7	Contour plot of the maximum shear stress around a Mode II crack.	301
7.8	The displacement of a Mode II crack under increasing shear stress.	303
7.9	A Mode I crack.	304
7.10	Contour plot of the maximum shear stress around a Mode I crack.	306
7.11	The displacement of a Mode I crack under increasing normal stress.	307
7.12	Solution for the contact between a string and a level surface.	310
7.13	Three candidate solutions for a contact problem.	311
7.14	The contact between a beam and a horizontal surface under a uniform pressure.	314
7.15	Contact between a rigid body and an elastic half-space.	317
7.16	The penetration of a quadratic punch into an elastic half-space.	319
7.17	A flexible ruler flattened against a table.	326
7.18	A wave travelling along a rope on the ground.	326
8.1	A typical stress–strain relationship for a plastic material.	329
8.2	The stress–strain relationship for a perfectly plastic material.	330
8.3	The forces acting on a particle at the surface of a granular material.	331
8.4	The normal force and frictional force acting on a surface element inside a granular material.	332
8.5	The Mohr circle.	333
8.6	The triaxial stress factor versus angle of friction.	336
8.7	An antiplane cut-and-weld operation.	339
8.8	The displacement field in an edge dislocation.	340
8.9	An edge dislocation in a square crystal lattice.	341
8.10	A moving edge dislocation.	342
8.11	The normalised torque versus twist applied to an elastic-plastic cylindrical bar.	347
8.12	The normalised torque versus twist applied to an elastic-plastic cylindrical bar, showing the recovery phase.	348

*List of illustrations*

xi

8.13	The free-boundary problem for an elastic-perfectly plastic torsion bar.	349
8.14	Residual shear stress in a gun barrel versus radial distance for different values of the maximum internal pressurisation.	352
8.15	The Tresca yield surface.	355
8.16	The von Mises yield surface.	356
8.17	The Coulomb yield surface.	358
8.18	Lüders bands in a thin sheet of metal.	369
8.19	The Mohr surface for three-dimensional granular flow.	373
8.20	The normalised torque versus twist applied to an elastic-plastic cylindrical bar undergoing a loading cycle.	375
9.1	(a) A spring; (b) a dashpot; (c) a spring and dashpot connected in parallel; (d) a spring and dashpot connected in series.	380
9.2	(a) Applied tension as a function of time. (b) Resultant displacement of a linear elastic spring. (c) Resultant displacement of a linear dashpot.	381
9.3	Displacement of a Voigt element due to the applied tension shown in Figure 9.2(a).	382
9.4	Displacement of a Maxwell element due to the applied tension shown in Figure 9.2(a).	383
9.5	(a) The variation of Young's modulus with position in a bar. (b) The corresponding longitudinal displacement.	392
9.6	(a) The variation of Young's modulus with position in a bar. (b) The corresponding longitudinal displacement.	395
9.7	A periodic microstructured shear modulus.	396
9.8	A symmetric, piecewise constant shear modulus distribution.	400
9.9	Some modulus distributions that are antisymmetric about the diagonals of a square.	402
9.10	Dimensionless wavenumber versus the Young's modulus non-uniformity parameter.	407
9.11	The one-dimensional squeezing of a sponge.	411
9.12	Dimensionless stress applied to a sponge versus dimensionless time for different values of the Péclet number.	412
9.13	A Jeffreys viscoelastic element.	418
9.14	A system of masses connected by springs and dashpots in parallel.	418
9.15	A system of masses connected by springs and dashpots in series.	419
9.16	Dimensionless wavenumber versus Young's modulus contrast for a piecewise uniform bar.	424
A1.1	A small reference box.	432
A1.2	Cylindrical polar coordinates.	437
A1.3	Spherical polar coordinates.	438

## Prologue

Although solid mechanics is a vitally important branch of applied mechanics, it is often less popular, at least among students, than its close relative, fluid mechanics. Several reasons can be advanced for this disparity, such as the prevalence of tensors in models for solids or the especial difficulty of handling nonlinearity. Perhaps the most daunting prospect for the student is the multitude of different behaviours that can occur and cause elementary theories of elasticity to become irrelevant in practice. Examples include fracture, buckling and plasticity, and these pose intellectual challenges in solid mechanics that are every bit as fascinating as concepts like flight, shock waves and turbulence in fluid dynamics. Our principal objective in this book is to demonstrate this fact to undergraduate and beginning graduate students.

We aim to give the subject as wide an accessibility as possible to mathematically-minded students and to emphasise the interesting mathematical issues that it raises. We do this by relating the theory to practical applications where surprising phenomena occur and where innovative mathematical methods are needed.

Our layout is essentially pragmatic. Although more advanced texts in solid mechanics often begin with quite general theories founded on basic mechanical and thermodynamic principles, we start from the very simplest models, based on elementary observations in engineering and physics, and build our way towards models that are the basis for current applied research in solid mechanics. Hence, we begin by deriving the basic Navier equations of linear elasticity, before illustrating the mathematical techniques that allow these equations to be solved in many different practically relevant situations, both static and dynamic. We then proceed to describe some approximate theories for the elastic deformation of thin solids, namely bars, strings, beams, rods, plates and shells. We soon discover that many everyday phenomena, such as the buckling of a beam under a compressive load, cannot be fully described

using linear theories. We therefore give a brief exposition of the general theory of nonlinear elasticity, and then show how formal asymptotic methods allow simplified linear and weakly nonlinear models to be systematically deduced. Although we regard such asymptotic techniques as invaluable to any applied mathematician, these last two topics may both be omitted on a first reading without loss of continuity. We go on to present simple models for fracture and contact, comparing and contrasting these apparently similar phenomena. Next, we show how plasticity theory can be used to describe situations where a solid yields under a sufficiently high stress. Finally, we show how elasticity theory may be generalised to include further physical effects, such as thermal stresses, viscoelasticity and porosity. These “combined fields” of solid mechanics are increasingly finding applications in industrial and medical processes, and pose ever more elaborate modelling questions.

Despite the breadth of the models and relevant techniques that will emerge in this book, we will usually try to present the theoretical developments *ab initio*. Nonetheless, the book is very far from being self-contained. Any student who aspires to becoming a solid mechanics specialist will have to delve further into the literature, and we will provide references to help with this.

We assume only that the reader has a reasonable familiarity with the calculus of several variables. Fluency with the more advanced techniques required for Chapters 6 and 7, in particular, will readily be acquired by a student who works through the exercises in the early Chapters, especially those cited in the text. Indeed, we firmly believe that solid mechanics provides a wonderful arena in which to build an understanding of such important mathematical areas as linear algebra, partial differential equations, complex variable theory, differential geometry and the calculus of variations. Our hope is that, having read this book, a student should be able to confront any practical problem that may be encountered in everyday solid mechanics with at least some idea of the basic mathematical modelling that will be required.

During the writing of this book, we received a great deal of help and inspiration as a result of discussions with David Allwright, Jon Chapman, Sam Howison, L. Mahadevan, Roman Novokshanov and Domingo Salazar, as well as many other colleagues and students too numerous to thank individually. We would like to express our particular gratitude to Gareth Jones, Hilary Ockendon and Tom Witelski who gave invaluable advice on draft Chapters. We are also indebted to David Tranah and his colleagues at Cambridge University Press for helping to make this book a reality.