1
Modelling solids

1.1 Introduction

In everyday life we regularly encounter physical phenomena that apparently vary continuously in space and time. Examples are the bending of a paper clip, the flow of water or the propagation of sound or light waves. Such phenomena can be described mathematically, to lowest order, by a *continuum model*, and this book will be concerned with that class of continuum models that describes solids. Hence, at least to begin with, we will avoid all consideration of the "atomistic" structure of solids, even though these ideas lead to great practical insight and also to some beautiful mathematics. When we refer to a solid "particle", we will be thinking of a very small region of matter but one whose dimension is nonetheless much greater than an atomic spacing.

For our purposes, the diagnostic feature of a solid is the way in which it responds to an applied system of forces and moments. There is no hard-and-fast rule about this but, for most of this book, we will say that a continuum is a *solid* when the response consists of *displacements* distributed through the material. In other words, the material starts at some *reference state*, from which it is displaced by a distance that depends on the applied forces. This is in contrast with a fluid, which has no special rest state and responds to forces via a *velocity* distribution. Our modelling philosophy is straightforward. We take the most fundamental pieces of experimental evidence, for example Hooke's law, and use mathematical ideas to combine this evidence with the basic laws of mechanics to construct a model that describes the elastic deformation of a continuous solid. Following this simple approach, we will find that we can construct solid mechanics theories for phenomena as diverse as earthquakes, ultrasonic testing and the buckling of railway tracks.

By basing our theory on Hooke's law, the simplest model of elasticity, for small enough forces and displacements, we will first be led to a system of differential equations that is both *linear*, and therefore mathematically tractable, and *reversible* for time-dependent problems. By this we mean that, when forces and moments are applied and then removed, the system eventually returns to its original state without any significant energy being lost, i.e. the system is not *dissipative*.

Reversibility may apply even when the forces and displacements are so large that the problem ceases to be linear; a rubber band, for example, can undergo large displacements and still return to its initial state. However, *nonlinear elasticity* encompasses some striking new behaviours not predicted by linear theory, including the possibility of multiple steady states and *buckling*. For many materials, experimental evidence reveals that even more dramatic changes can take place as the load increases, the most striking phenomenon being that of *fracture* under extreme stress. On the other hand, as can be seen by simply bending a metal paper clip, irreversibility can readily occur and this is associated with *plastic* flow that is significantly dissipative. In this situation, the solid takes on some of the attributes of a fluid, but the model for its flow is quite different from that for, say, water.

Practical solid mechanics encompasses not only all the phenomena mentioned above but also the effects of elasticity when combined with heat transfer (leading to thermoelasticity) and with genuine fluid effects, in cases where the material flows even in the absence of large applied forces (leading to viscoelasticity) or when the material is porous (leading to poroelasticity). We will defer consideration of all these combined fields until the final chapter.

1.2 Hooke's law

Robert Hooke (1678) wrote

"it is . . . evident that the rule or law of nature in every springing body is that the force or power thereof to restore itself to its natural position is always proportionate to the distance or space it is removed therefrom, whether it be by rarefaction, or separation of its parts the one from the other, or by condensation, or crowding of those parts nearer together."

Hooke's observation is exemplified by a simple high-school physics experiment in which a tensile force T is applied to a spring whose natural length is L. *Hooke's law* states that the resulting extension of the spring is proportional to T: if the new length of the spring is ℓ, then

$$T = k(\ell - L), \tag{1.2.1}$$

where the constant of proportionality k is called the spring constant.

Hooke devised his law while designing clock springs, but noted that it appears to apply to all "*springy bodies whatsoever, whether metal, wood, stones, baked earths, hair, horns, silk, bones, sinews, glass and the like.*" In practice, it is commonly observed that k scales with $1/L$; that is, everything else being equal, a sample that is initially twice as long will stretch twice as far under the same force. It is therefore sensible to write (1.2.1) in the form

$$T = k' \frac{\ell - L}{L}, \tag{1.2.2}$$

where k' is the *elastic modulus* of the spring, which will be defined more rigorously in Chapter 2. The dimensionless quantity $(\ell - L)/L$, measuring the extension relative to the initial length, is called the *strain*.

Equation (1.2.2) is the simplest example of the all-important *constitutive law* relating the force to displacement. As shown in Exercise 1.3, it is possible to construct a one-dimensional continuum model for an elastic solid from this law, but, to generalise it to a three-dimensional continuum, we first need to generalise the concepts of strain and tension.

1.3 Lagrangian and Eulerian coordinates

Suppose that a three-dimensional solid starts, at time $t = 0$, in its rest state, or *reference state*, in which no macroscopic forces exist in the solid or on its boundary. Under the action of any subsequently applied forces and moments, the solid will be deformed such that, at some later time t, a "particle" in the solid whose initial position was the point $\boldsymbol{X}$ is *displaced* to the point $\boldsymbol{x}(\boldsymbol{X}, t)$. This is a *Lagrangian* description of the continuum: if the independent variable $\boldsymbol{X}$ is held fixed as t increases, then $\boldsymbol{x}(\boldsymbol{X}, t)$ labels a material particle. In the alternative *Eulerian* approach, we consider the material point which currently occupies position $\boldsymbol{x}$ at time t, and label its initial position by $\boldsymbol{X}(\boldsymbol{x}, t)$. In short, the Eulerian coordinate $\boldsymbol{x}$ is *fixed in space*, while the Lagrangian coordinate $\boldsymbol{X}$ is *fixed in the material*.

The *displacement* $\boldsymbol{u}(\boldsymbol{X}, t)$ is defined in the obvious way to be the difference between the current and initial positions of a particle, that is

$$\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X}. \tag{1.3.1}$$

Many basic problems in solid mechanics amount to determining the displacement field $\boldsymbol{u}$ corresponding to a given system of applied forces.

The mathematical consequence of our statement that the solid is a continuum is that there must be a smooth one-to-one relationship between $\boldsymbol{X}$ and $\boldsymbol{x}$, *i.e.* between any particle's initial position and its current position.

This will be the case provided the Jacobian of the transformation from $\boldsymbol{X}$ to $\boldsymbol{x}$ is bounded away from zero:

$$0 < J < \infty, \qquad \text{where} \quad J = \det\left(\frac{\partial x_i}{\partial X_j}\right). \tag{1.3.2}$$

The physical significance of J is that it measures the change in a small volume compared with its initial volume:

$$\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = J\,\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3, \qquad \text{or} \quad \mathrm{d}\boldsymbol{x} = J\,\mathrm{d}\boldsymbol{X} \tag{1.3.3}$$

as shorthand. The positivity of J means that we exclude the possibility that the solid turns itself inside-out.

We can use (1.3.3) to derive a kinematic equation representing conservation of mass. Consider a moving volume $V(t)$ that is always bounded by the same solid particles. Its mass at time t is given, in terms of the density $\rho(\boldsymbol{X}, t)$, by

$$M(t) = \iiint_{V(t)} \rho\,\mathrm{d}\boldsymbol{x} = \iiint_{V(0)} \rho J\,\mathrm{d}\boldsymbol{X}. \tag{1.3.4}$$

Since $V(t)$ designates a fixed set of material points, $M(t)$ must be a constant, namely its initial value $M(0)$:

$$\iiint_{V(0)} \rho J\,\mathrm{d}\boldsymbol{X} = M(t) = M(0) = \iiint_{V(0)} \rho_0\,\mathrm{d}\boldsymbol{X}, \tag{1.3.5}$$

where ρ_0 is the density in the rest state. Since V is arbitrary, we deduce that

$$\rho J = \rho_0. \tag{1.3.6}$$

Hence, we can calculate the density at any time t in terms of ρ_0 and the displacement field. The initial density ρ_0 is usually taken as constant, but (1.3.6) also applies if $\rho_0 = \rho_0(\boldsymbol{X})$.

1.4 Strain

To generalise the concept of strain introduced in Section 1.2, we consider the deformation of a small line segment joining two neighbouring particles with initial positions $\boldsymbol{X}$ and $\boldsymbol{X} + \delta\boldsymbol{X}$. At some later time, the solid deforms such that the particles are displaced to $\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t)$ and $\boldsymbol{X} + \delta\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X} + \delta\boldsymbol{X}, t)$ respectively. Thus we can use Taylor's theorem to show that the line element

$\delta \boldsymbol{X}$ that joins the two particles is transformed to

$$\delta \boldsymbol{x} = \delta \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X} + \delta \boldsymbol{X}, t) - \boldsymbol{u}(\boldsymbol{X}, t) = \delta \boldsymbol{X} + (\delta \boldsymbol{X} \cdot \boldsymbol{\nabla})\boldsymbol{u}(\boldsymbol{X}, t) + \cdots, \quad (1.4.1)$$

where

$$(\delta \boldsymbol{X} \cdot \boldsymbol{\nabla}) = \delta X_1 \frac{\partial}{\partial X_1} + \delta X_2 \frac{\partial}{\partial X_2} + \delta X_3 \frac{\partial}{\partial X_3}. \quad (1.4.2)$$

Let $L = |\delta \boldsymbol{X}|$ and $\ell = |\delta \boldsymbol{x}|$ denote the initial and current lengths respectively of the line segment; the difference $\ell - L$ is known as the *stretch*. Then, to lowest order in L,

$$\ell^2 = |\delta \boldsymbol{X} + (\delta \boldsymbol{X} \cdot \boldsymbol{\nabla})\boldsymbol{u}(\boldsymbol{X}, t)|^2. \quad (1.4.3)$$

Although we will try in subsequent chapters to minimise the use of suffices, it is helpful at this stage to introduce components so that $\boldsymbol{X} = (X_i) = (X_1, X_2, X_3)^{\mathrm{T}}$ and similarly for $\boldsymbol{u}$. Then (1.4.3) may be written in the form

$$\ell^2 - L^2 = 2 \sum_{i,j=1}^{3} \mathcal{E}_{ij}\, \delta X_i \delta X_j, \quad (1.4.4)$$

where

$$\mathcal{E}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right). \quad (1.4.5)$$

By way of introduction to some notation that will be useful later, we point out that (1.4.4) may be written in at least two alternative ways. First, we may invoke the *summation convention*, in which one automatically sums over any repeated suffix. This avoids the annoyance of having to write explicit summation, so (1.4.4) is simply

$$\ell^2 = L^2 + 2\mathcal{E}_{ij}\, \delta X_i \delta X_j, \quad \text{where} \quad \mathcal{E}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right). \quad (1.4.6)$$

Second, we note that $\ell^2 - L^2$ is a quadratic form on the symmetric matrix $\mathcal{E}$ whose components are $(\mathcal{E}_{ij})$:

$$\ell^2 - L^2 = 2\, \delta \boldsymbol{X}^{\mathrm{T}} \mathcal{E}\, \delta \boldsymbol{X}. \quad (1.4.7)$$

It is clear from (1.4.4) that the stretch is measured by the quantities $\mathcal{E}_{ij}$; in particular, the stretch is zero for all line elements if and only if $\mathcal{E}_{ij} \equiv 0$. It is thus natural to identify $\mathcal{E}_{ij}$ with the *strain*. Now let us ask: "what happens when we perform the same calculation in a coordinate system rotated by an orthogonal matrix $P = (p_{ij})$?" Intuitively, we might expect the strain to be invariant under such a rotation, and we can verify that this is so as follows.

The vectors $\boldsymbol{X}$ and $\boldsymbol{u}$ are transformed to $\boldsymbol{X}'$ and $\boldsymbol{u}'$ in the new coordinate system, where

$$\boldsymbol{X}' = P\boldsymbol{X}, \qquad \boldsymbol{u}' = P\boldsymbol{u}. \tag{1.4.8}$$

Since P is orthogonal, (1.4.8) may be inverted to give $\boldsymbol{X} = P^{\mathrm{T}}\boldsymbol{X}'$. Alternatively, using suffix notation, we have

$$X_\beta = p_{j\beta}X'_j, \qquad u'_i = p_{i\alpha}u_\alpha. \tag{1.4.9}$$

The strain in the new coordinate system is denoted by

$$\mathcal{E}'_{ij} = \frac{1}{2}\left(\frac{\partial u'_i}{\partial X'_j} + \frac{\partial u'_j}{\partial X'_i} + \frac{\partial u'_k}{\partial X'_i}\frac{\partial u'_k}{\partial X'_j}\right), \tag{1.4.10}$$

which may be manipulated using the chain rule, as shown in Exercise 1.4, to give

$$\mathcal{E}'_{ij} = p_{i\alpha}p_{j\beta}\mathcal{E}_{\alpha\beta}. \tag{1.4.11}$$

In matrix notation, (1.4.11) takes the form

$$\mathcal{E}' = P\mathcal{E}P^{\mathrm{T}}, \tag{1.4.12}$$

so the 3×3 symmetric array $(\mathcal{E}_{ij})$ transforms exactly like a matrix representing a linear transformation of the vector space $\mathbb{R}^3$. Arrays that obey the transformation law (1.4.11) are called *second-rank Cartesian tensors*, and $\mathcal{E} = (\mathcal{E}_{ij})$ is therefore called the *strain tensor*.†

Almost as important as the fact that $\mathcal{E}$ is a tensor is the fact that it can vanish without $\boldsymbol{u}$ vanishing. More precisely, if we consider a rigid-body translation and rotation

$$\boldsymbol{u} = \boldsymbol{c} + (Q - \mathcal{I})\boldsymbol{X}, \tag{1.4.13}$$

where $\mathcal{I}$ is the identity matrix while the vector $\boldsymbol{c}$ and orthogonal matrix Q are constant, then $\mathcal{E}$ is identically zero. This result follows directly from substituting (1.4.13) into (1.4.6) and using the fact that $QQ^{\mathrm{T}} = \mathcal{I}$, and confirms our intuition that a rigid-body motion induces no deformation.

† The word "tensor" as used here is effectively synonymous with "matrix", but it is easy to generalise (1.4.11) to a tensor with any number of indices. A vector, for example, is a tensor with just one index.

1.5 Stress

In the absence of any volumetric (e.g. gravitational or electromagnetic) effects, a force can only be transmitted to a solid by being applied to its boundary. It is, therefore, natural to consider the force per unit area or *stress* applied at that boundary. To do so, we now analyse an infinitesimal surface element, whose area and unit normal are $\mathrm{d}a$ and $\boldsymbol{n}$ respectively. If it is contained within a stressed medium, then the material on (say) the side into which $\boldsymbol{n}$ points will exert a force $\mathrm{d}\boldsymbol{f}$ on the element. (By Newton's third law, the material on the other side will also exert a force equal to $-\mathrm{d}\boldsymbol{f}$.) In the expectation that the force should be proportional to the area $\mathrm{d}a$, we write

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{\sigma}\,\mathrm{d}a, \tag{1.5.1}$$

where $\boldsymbol{\sigma}$ is called the *traction* or *stress* acting on the element.

Perhaps the most familiar example is that of an inviscid fluid, in which the stress is related to the pressure p by

$$\boldsymbol{\sigma} = -p\boldsymbol{n}. \tag{1.5.2}$$

This expression implies that (i) the stress acts only in a direction normal to the surface element, (ii) the magnitude of the stress (i.e. p) is independent of the direction of $\boldsymbol{n}$. In an elastic solid, neither of these simplifying assumptions holds; we must allow for stress which acts in both tangential and normal directions and whose magnitude depends on the orientation of the surface element.

First consider a surface element whose normal points in the x_1-direction, and denote the stress acting on such an element by $\boldsymbol{\tau}_1 = (\tau_{11}, \tau_{21}, \tau_{31})^{\mathrm{T}}$. By doing the same for elements with normals in the x_2- and x_3-directions, we generate three vectors $\boldsymbol{\tau}_j$ $(j = 1, 2, 3)$, each representing the stress acting on an element normal to the x_j-direction. In total, therefore, we obtain nine scalars τ_{ij} $(i, j = 1, 2, 3)$, where τ_{ij} is the i-component of $\boldsymbol{\tau}_j$, that is

$$\boldsymbol{\tau}_j = \tau_{ij}\boldsymbol{e}_i, \tag{1.5.3}$$

where $\boldsymbol{e}_i$ is the unit vector in the x_i-direction.

The scalars τ_{ij} may be used to determine the stress on an arbitrary surface element by considering the tetrahedron shown in Figure 1.1. Here a_i denotes the area of the face orthogonal to the x_i-axis. The fourth face has area $a = \sqrt{a_1^2 + a_2^2 + a_3^2}$; in fact if this face has unit normal $\boldsymbol{n}$ as shown, with components (n_i), then it is an elementary exercise in trigonometry to show that $a_i = an_i$.

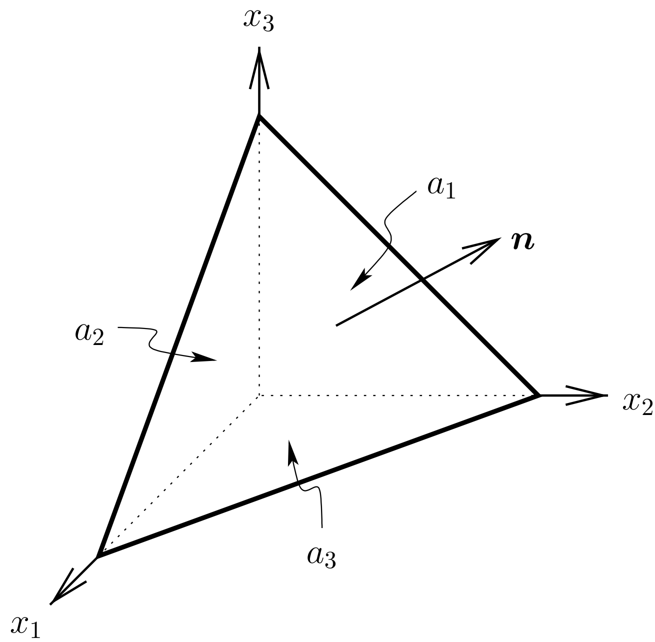

Fig. 1.1 A reference tetrahedron; a_i is the area of the face orthogonal to the x_i-axis.

The *outward* normal to the face with area a_1 is in the *negative* x_1-direction and the force on this face is thus $-a_1\boldsymbol{\tau}_1$. Similar expressions hold for the faces with areas a_2 and a_3. Hence, if the stress on the fourth face is denoted by $\boldsymbol{\sigma}$, then the total force on the tetrahedron is

$$\boldsymbol{f} = a\boldsymbol{\sigma} - a_j\boldsymbol{\tau}_j. \tag{1.5.4}$$

When we substitute for a_j and $\boldsymbol{\tau}_j$, we find that the components of $\boldsymbol{f}$ are given by

$$f_i = a\left(\sigma_i - \tau_{ij}n_j\right). \tag{1.5.5}$$

Now we shrink the tetrahedron to zero volume. Since the area a scales with ℓ^2, where ℓ is a typical edge length, while the volume is proportional to ℓ^3, if we apply Newton's second law and insist that the acceleration be finite, we see that $\boldsymbol{f}/a$ must tend to zero as $\ell \to 0$.† Hence we deduce an

† Readers of a sensitive disposition may be slightly perturbed by our glibly letting the *dimensional* variable ℓ tend to zero: if ℓ is reduced indefinitely then we will eventually reach an atomic scale on which the solid can no longer be treated as a continuum. We reassure such readers that (1.5.6) can be more rigorously justified provided the macroscopic dimensions of the solid are large compared to any atomistic length-scale.

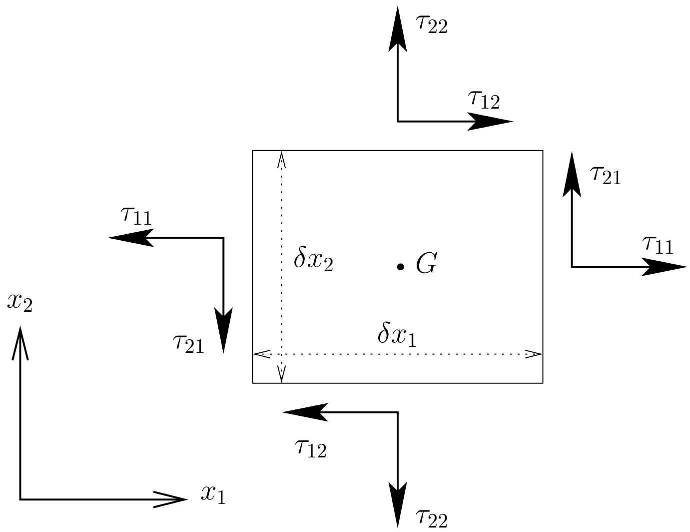

Fig. 1.2 The forces acting on a small two-dimensional element.

expression for $\boldsymbol{\sigma}$:

$$\sigma_i = \tau_{ij} n_j, \qquad \text{or} \qquad \boldsymbol{\sigma} = \tau \boldsymbol{n}. \tag{1.5.6}$$

This important result enables us to find the stress on *any* surface element in terms of the nine quantities $(\tau_{ij}) = \tau$.

Now let us follow Section 1.4 and examine what happens to τ_{ij} when we rotate the axes by an orthogonal matrix P. In the new frame, (1.5.6) will become

$$\boldsymbol{\sigma}' = \tau' \boldsymbol{n}' \tag{1.5.7}$$

where, since $\boldsymbol{\sigma}$ and $\boldsymbol{n}$ are vectors, they transform according to

$$\boldsymbol{\sigma}' = P\boldsymbol{\sigma}, \qquad \boldsymbol{n}' = P\boldsymbol{n}. \tag{1.5.8}$$

It follows that $\tau'\boldsymbol{n} = (P\tau P^{\mathrm{T}})\boldsymbol{n}$ and so, since $\boldsymbol{n}$ is arbitrary,

$$\tau' = P\tau P^{\mathrm{T}}, \qquad \text{or} \qquad \tau'_{ij} = p_{i\alpha} p_{j\beta} \tau_{\alpha\beta}. \tag{1.5.9}$$

Thus τ_{ij}, like $\mathcal{E}_{ij}$, is a second-rank tensor, called the *Cauchy stress tensor*.

We can make one further observation about τ_{ij} by considering the angular momentum of the small two-dimensional solid element shown in Figure 1.2. The net anticlockwise moment acting about the centre of mass G is (per unit length in the x_3-direction)

$$2\,(\tau_{21}\delta x_2)\,\frac{\delta x_1}{2} - 2\,(\tau_{12}\delta x_1)\,\frac{\delta x_2}{2},$$

where τ_{21} and τ_{12} are evaluated at G to lowest order. By letting the rectangle shrink to zero (see again the footnote on page 8), and insisting that the angular acceleration be finite, we deduce that $\tau_{12} = \tau_{21}$. This argument can be generalised to three dimensions (see Exercise 1.5) and it shows that

$$\tau_{ij} \equiv \tau_{ji} \tag{1.5.10}$$

for all i and j, i.e. that τ_{ij}, like $\mathcal{E}_{ij}$, is a symmetric tensor.

1.6 Conservation of momentum

Now we derive the basic governing equation of solid mechanics by applying Newton's second law to a material volume $V(t)$ that moves with the deforming solid:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \frac{\partial u_i}{\partial t} \rho \,\mathrm{d}\boldsymbol{x} = \iiint_{V(t)} g_i \rho \,\mathrm{d}\boldsymbol{x} + \iint_{\partial V(t)} \tau_{ij} n_j \,\mathrm{d}a. \tag{1.6.1}$$

The terms in (1.6.1) represent successively the rate of change of momentum of the material in $V(t)$, the force due to an external body force $\boldsymbol{g}$, such as gravity, and the traction exerted on the boundary of V, whose unit normal is $\boldsymbol{n}$, by the material around it. We differentiate under the integral (using the fact that $\rho \,\mathrm{d}\boldsymbol{x} = \rho_0 \,\mathrm{d}\boldsymbol{X}$ is independent of t) and apply the divergence theorem to the final term to obtain

$$\iiint_{V(t)} \frac{\partial^2 u_i}{\partial t^2} \rho \,\mathrm{d}\boldsymbol{x} = \iiint_{V(t)} g_i \rho \,\mathrm{d}\boldsymbol{x} + \iiint_{V(t)} \frac{\partial \tau_{ij}}{\partial x_j} \,\mathrm{d}\boldsymbol{x}. \tag{1.6.2}$$

Assuming each integrand is continuous, and using the fact that $V(t)$ is arbitrary, we arrive at *Cauchy's momentum equation*:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}. \tag{1.6.3}$$

This may alternatively be written in vector form by adopting the following notation for the divergence of a tensor: we define the ith component of $\boldsymbol{\nabla} \cdot \tau$ to be

$$(\boldsymbol{\nabla} \cdot \tau)_i = \frac{\partial \tau_{ji}}{\partial x_j}. \tag{1.6.4}$$

Since τ is symmetric, we may thus write Cauchy's equation as

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho \boldsymbol{g} + \boldsymbol{\nabla} \cdot \tau. \tag{1.6.5}$$

This equation applies to *any* continuous medium for which a displacement $\boldsymbol{u}$ and stress tensor τ can be defined. The distinction between solid, fluid