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94

The volume of convex bodies and Banach space geometry

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The volume of convex bodies and Banach space geometry

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Contents

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Gilles Pisier
Frontmatter
More information

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Gilles Pisier
Frontmatter
More information

Introduction

The aim of these notes is to give a self-contained presentation of a number of recent results (mainly obtained during the period 1984–86) which relate the volume of convex bodies in $\mathbf{R}^n$ and the geometry of the corresponding finite-dimensional normed spaces.

The methods we employ combine traditional ideas in the Theory of Convex Sets (maximal volume ellipsoids, mixed volumes) together with Probability (Gaussian processes), Approximation Theory (Entropy numbers), and the Local Theory of Banach spaces (K-convexity, cotype and type).

During the last decade, considerable progress was achieved in the *Local Theory*, i.e. the part of Banach Space Theory which uses finite-dimensional (f.d. for short) tools to study infinite dimensional spaces.

The paper [FLM] was one of the first to relate the analytic properties of a Banach space (such as type and cotype) with the geometry of its f.d. subspaces. More precisely, let $B \subset \mathbf{R}^n$ be a ball (by this we mean a compact convex symmetric body admitting the origin as an interior point). In [FLM] the following question is studied:

Fix $\varepsilon > 0$, for which integer k does there exist a $(1+\varepsilon)$-Euclidean section of B? Equivalently, for which k does there exist a subspace $F \subset \mathbf{R}^n$ with dimension k and an ellipsoid $D \subset F$ such that

$$D \subset B \cap F \subset (1+\varepsilon)D? \tag{0.1}$$

By a celebrated result of Dvoretzky, this is *always* true for $k = k(\varepsilon, n)$, with $k(\varepsilon, n) \to \infty$ when $n \to \infty$ (with $\varepsilon > 0$ fixed). More precisely ([M6]), this holds for $k = [\phi(\varepsilon) \operatorname{Log} n]$, with $\phi(\varepsilon) > 0$ depending only on ε. This estimate is the best possible in general; indeed, where B is the unit cube $B = [-1,1]^n$ (i.e. the unit ball of ℓ_∞^n) then this $\operatorname{Log} n$ bound cannot be improved. One of the main discoveries of [FLM] is that if B and all its sections are far from cubes, in a certain analytic sense, then the estimate can be improved to

$$k(\varepsilon, n) = [\phi(\varepsilon) n^\alpha]$$

for some $\alpha > 0$.

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Gilles Pisier
Frontmatter
More information

Precisely, if the normed space E for which B is the unit ball has cotype $q(2 \le q < \infty)$ with constant C then this holds with $\alpha = 2/q$ and $\phi(\varepsilon)$ depending only on ε and C.

The most striking case is the case $q = 2$ (hence $\alpha = 1$) for which we find in answer to (0.1) an integer k proportional to n. For instance, this case covers the case of $E = \ell_p^n$ for $1 \le p < 2$. By duality, (0.1) implies that B°, the polar of the ball B, admits a projection onto F which is $(1+\varepsilon)$-equivalent to an ellipsoid. Namely, if P_F is the orthogonal projection from $\mathbf{R}^n$ onto F, we have

$$(1+\varepsilon)^{-1} D^\circ \subset P_F(B^\circ) \subset D^\circ. \tag{0.2}$$

Since B is arbitrary, we can replace B by B° in (0.2). Thus, Dvoretzky's Theorem says that any n-dimensional body B admits both k-dimensional sections and k-dimensional projections which are almost ellipsoids with $k \to \infty$ when $n \to \infty$. However, in general k must remain small compared with n.

One of the striking recent discoveries of Milman [M1] is that if one considers the class of all projections of sections of B (instead of either projections or sections) then one can *always* find a projection of a section of B (say $P_{F_2}(F_1 \cap B)$ with $F_2 \subset F_1 \subset \mathbf{R}^n$) which is $(1+\varepsilon)$-equivalent to an ellipsoid and has dimension $k = [\psi(\varepsilon) n]$ with $\psi(\varepsilon) > 0$ depending only on $\varepsilon > 0$.

Thus we find again k proportional to n, but this time without any assumption on B.

This surprising result gives the impression that in a number of questions an arbitrary ball in $\mathbf{R}^n$ should behave essentially like an ellipsoid. We will see several illustrations of this in the present volume. We now describe the contents.

This book has two distinct parts.

The objective of the first part (Chapters 1 to 8) is to present self-contained proofs of three fundamental results:

(I) *The quotient of subspace Theorem (Q.S.-Theorem for short) due to Milman* [M1]*: For each $0 < \delta < 1$ there is a constant $C = C(\delta)$ such that every n-dimensional normed space E admits a quotient of a subspace $F = E_1/E_2$ (with $E_2 \subset E_1 \subset E$) with dimension $\dim F \ge \delta n$ which is C-isomorphic to a Euclidean space.*

(II) *The inverse Santaló inequality due to Bourgain and Milman* [BM]*: There are positive constants α and β (independent of n) such that for all balls $B \subset \mathbf{R}^n$ we have*

$$\alpha/n \le (\mathrm{vol}(B)\,\mathrm{vol}(B^\circ))^{1/n} \le \beta/n.$$

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Gilles Pisier
Frontmatter
More information

(The upper bound goes back to a 1949 article by Santaló [Sa1].)

(III) *The inverse Brunn–Minkowski inequality due to Milman [M5]: Two balls B_1, B_2 in $\mathbf{R}^n$ can always be transformed (by a volume preserving linear isomorphism) into balls $\widetilde{B}_1, \widetilde{B}_2$ which satisfy*

$$\mathrm{vol}(\widetilde{B}_1+\widetilde{B}_2)^{1/n} \le C\left[\mathrm{vol}(\widetilde{B}_1)^{1/n}+\mathrm{vol}(\widetilde{B}_2)^{1/n}\right]$$

where C is a numerical constant independent of n. Moreover, the polars $\widetilde{B}_1^\circ, \widetilde{B}_2^\circ$ and all their multiples also satisfy a similar inequality.

We present *two different approaches* to these results which can be read essentially independently. In Chapter 7, we reverse the chronological order; we prove (III) first and then deduce (II) and (I) as easy consequences.

In Chapter 8 we prove (I) by essentially the original method of [M1] and then give a very simple proof that (I) implies (II).

In the second part of the book (Chapters 10 to 15) we give a detailed exposition focused on the recently introduced classes of Banach spaces of weak cotype 2 or of weak type 2 and the intersection of these classes, the class of weak Hilbert spaces.

The previous chapters contain complete proofs of all the necessary ingredients for these results and various related estimates. Let us now review the contents of this book, chapter by chapter.

Chapter 1 introduces some terminology, notation, and preliminary background.

In Chapter 2 one of our main tools—the majorization of the Gaussian K-convexity constant of an n-dimensional space—is discussed in detail.

The connections between Gaussian measures and volume estimates are numerous. For instance, consider the following classical formula where B is any ball in $\mathbf{R}^n$, γ_n the canonical probability measure on $\mathbf{R}^n$ and D the Euclidean unit ball in $\mathbf{R}^n$:

$$\lim_{t\to\infty}\frac{\mathrm{vol}(B+tD)-\mathrm{vol}(tD)}{t^{n-1}\,\mathrm{vol}(D)} = c_n n^{1/2}\int \sup_{t\in B} < t, x > d\gamma_n(x), \tag{0.3}$$

where c_n is a constant tending to 1 when $n\to\infty$ (see Remark 1.5 and the beginning of Chapter 9 for more information). We will see more connections when we come to Chapter 5.

In Chapter 3 we present several properties of the maximal volume ellipsoids. This goes back to Fritz John who proved in a 1948 paper

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Gilles Pisier
Frontmatter
More information

[Joh] that any ball $B \subset \mathbf{R}^n$ contains a unique ellipsoid of maximal volume. As observed more recently by D. Lewis [L], one may impose on the class of ellipsoids D various different types of constraint (instead of $D \subset B$) and consider in each case the ellipsoid of maximal volume.

In particular, for a given ball B, we can consider the ellipsoid of maximal volume among all ellipsoids D such that the quantity (0.3) is ≤ 1. This ellipsoid (called the ℓ-ellipsoid in the sequel) plays a crucial role in the proofs of (I), (II), and (III) above.

In Chapter 4 we present the proof of Dvoretzky's Theorem and several related facts. We follow the usual *concentration of measure* approach (cf. [M6], [FLM]) but we use Gaussian measures instead of the Haar measure on the orthogonal group. This approach underlines the close connection between geometric characteristics of a space (here the dimension of the almost Euclidean sections of the unit ball) and Gaussian random variables.

In Chapter 5 we present results from the theory of Gaussian processes, mainly the Dudley–Sudakov Theorem (Theorems 5.6 and 5.5) which gives both a lower and an upper bound for integrals such as

$$\mathbf{E} \sup_{t \in B} X_t \tag{0.4}$$

when (X_t) is a Gaussian process indexed by a set B. These estimates are given in terms of the metric $d(s,t) = \|X_t - X_s\|_2$ on B and involve the smallest number of balls of d-radius ε which are enough to cover B. Such bounds for (0.4) can be related to volume estimates, for instance via (0.3).

These inequalities can be reformulated in the language of entropy numbers of compact linear operators. We present a brief introduction to the theory of these numbers in Chapter 5. In particular, several results due to B. Carl will be very useful in subsequent chapters.

In Chapter 6 we present the notion of volume ratio. The volume ratio of an n-dimensional space E with unit ball B is defined as

$$\left(\frac{\mathrm{vol}(B)}{\mathrm{vol}(D_{\max})}\right)^{1/n},$$

where $D_{\max} \subset B$ is the maximal volume ellipsoid. This was introduced by Szarek [S], [ST] following earlier work of Kašin [Ka1] in Approximation Theory. Szarek observed that the ℓ_1^n-balls have a volume ratio uniformly bounded (when $n \to \infty$) and that this implies the striking orthogonal decomposition of ℓ_1^{2n} (due to Kašin) as $E_1 + E_2$ with E_1, E_2

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Gilles Pisier
Frontmatter
More information

both uniformly isomorphic to ℓ_2^n. This is now often called the Kašin decomposition of ℓ_1^n. We present Szarek's proof of this in Chapter 6 as well as several properties of spaces with bounded volume ratio.

To give the flavor of what goes on in this book, we wish to record here several interesting inequalities about the volume ratio. Let us denote by $\| \ \|_B$ the gauge of B. Let B_2 be the canonical Euclidean ball with its normalized surface measure σ on the boundary. Note that we have for any norm on $\mathbf{R}^n$

$$\int \|x\|_B \, d\sigma(x) = c_n n^{-1/2} \int \|x\|_B \, d\gamma_n(x) \tag{0.5}$$

with c_n as above ($c_n \to 1$ when $n \to \infty$).

Integrating in polar coordinates, we have (see Chapter 6)

$$\left(\frac{\mathrm{vol}(B)}{\mathrm{vol}(B_2)}\right)^{1/n} = \left(\int \|x\|_B^{-n} \, d\sigma(x)\right)^{1/n}. \tag{0.6}$$

By a classical inequality of Urysohn (see Chapter 1), we have

$$\left(\int \|x\|_B^{-n} \, d\sigma(x)\right)^{1/n} \le \int \|x\|_{B^\circ} \, d\sigma(x). \tag{0.7}$$

On the other hand, by convexity we have, obviously,

$$\left(\int \|x\|_B \, d\sigma(x)\right)^{-1} \le \left(\int \|x\|_B^{-n} \, d\sigma(x)\right)^{1/n}. \tag{0.8}$$

These inequalities (0.5) to (0.8) point at another close connection between volume ratio and Gaussian integrals. In particular, if we denote

$$\bigwedge = \int \|x\|_B \, d\sigma(x) \cdot \int \|x\|_{B^\circ} \, d\sigma(x),$$

then we find

$$\frac{1}{\bigwedge} \le \left(\frac{\mathrm{vol}(B)\,\mathrm{vol}(B^\circ)}{\mathrm{vol}(B_2)^2}\right)^{1/n} \le \bigwedge.$$

Now a simple computation shows that

$$\int \|x\|_B^2 \, d\sigma(x) = n^{-1} \int \|x\|_B^2 \, d\gamma_n(x).$$

Therefore

$$\wedge \le n^{-1} \left(\int \|x\|_B^2 \, d\gamma_n(x) \cdot \int \|x\|_{B^\circ}^2 \, d\gamma_n(x)\right)^{1/2}.$$

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Gilles Pisier
Frontmatter
More information

But the product $\mathrm{vol}(B)\cdot\mathrm{vol}(B^\circ)$ is an "affine invariant", i.e. it does not change if we replace B by $u^{-1}(B)$ for any u in $GL(n)$ (we denote here by $GL(n)$ the set of all invertible linear transfomations on $\mathbf{R}^n$). Note that $\|x\|_{u^{-1}(B)} = \|ux\|_B$ and

$$\|x\|_{(u^{-1}B)^\circ} = \|u^{-1*}x\|_{B^\circ}.$$

Therefore, the preceding estimate can be rewritten as follows:

$$\frac{1}{\widetilde{\wedge}} \le \left(\frac{\mathrm{vol}(B)\,\mathrm{vol}(B^\circ)}{\mathrm{vol}(B_2)^2}\right)^{1/n} \le \widetilde{\wedge},$$

where

$$\widetilde{\wedge} = \inf_{u\in GL(n)} n^{-1}\left(\int \|ux\|_B^2 d\gamma_n(x)\int \|u^{-1*}x\|_{B^\circ}^2 d\gamma_n(x)\right)^{1/2}.$$

Equivalently, with the notation of Chapter 3, if E is the normed space which admits B as its unit ball, we have

$$\widetilde{\wedge} = \inf\left\{n^{-1}\ell(u)\ell(u^{-1*})\middle| u : \ell_2^n \to E \quad u \text{ invertible}\right\}.$$

This shows that (II) (and similarly (I) or (III)) are easy to derive when $\widetilde{\wedge}$ can be bounded above. This is possible in the K-convex case (see Chapter 3); namely, we prove in Chapter 3 that $\widetilde{\wedge} \le K(E)$ where $K(E)$ is the "K-convexity constant" of E, but unfortunately the constant $K(E)$ does not remain bounded when E and n are arbitrary. Nevertheless, the somewhat special properties of this constant $K(E)$ (as presented in Chapter 2) are among the most crucial tools in the proofs of (I), (II), or (III) above.

In Chapter 7 we present our proof of (III). The main ingredients are the results of Chapters 2, 3, 5, and 6.

In Chapter 8 we present Milman's proof of (I) and our proof of (II) as a simple consequence of (I). We also derive very directly from (II) the duality of entropy numbers for operators of rank n, due to König and Milman (see Theorem 8.10).

Both chapters can be read independently. In particular, the reader who wishes to read Chapter 8 before Chapter 7 should be warned that Chapter 7 is much simplified if one assumes (II) known. In particular, $M(B, D)$ reduces to just one of its factors if one already knows (II).

In Chapter 9 we study the volume numbers $v_n(T)$ of an operator $T : X \to \ell_2$. These numbers are defined as

$$v_n(T) = \sup\left(\frac{\mathrm{vol}(P(\overline{T(B_X)}))}{V_n}\right)^{1/n},$$

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Gilles Pisier
Frontmatter
More information

where the supremum runs over all orthogonal projections P of rank n on ℓ_2 and where we have denoted by V_n the volume of the canonical Euclidean unit ball.

We compare these numbers with the entropy numbers of T and show that they are almost equivalent. For instance, we show that if $\alpha < -1/2$ then

$$\limsup \frac{\operatorname{Log} v_n(T)}{\operatorname{Log} n} = \alpha \text{ iff } \limsup \frac{\operatorname{Log} e_n(T)}{\operatorname{Log} n} = \alpha.$$

This was conjectured by Dudley [Du1] and was established in [MP2]. The results of this chapter are not used in the sequel.

Chapters 10 to 15 constitute the second part of this book. They are motivated by the following property of certain Banach spaces X.

(*) *There is* $0 < \delta < 1$ *and a constant* C *such that every f.d. subspace* $E \subset X$ *contains a subspace* $F \subset E$ *with* $\dim F \geq \delta \dim E$ *which is* C*-isomorphic to a Euclidean space.*

As mentioned above, it was proved in [FLM] that this holds if X has cotype 2. Moreover, an example of Johnson (cf. [FLM]) shows that conversely (*) does not imply cotype 2, but only cotype $2+\varepsilon$ for every $\varepsilon > 0$.

Independently of [FLM] and almost simultaneously, Kăsin ([Ka1]) proved that (*) holds for $E = \ell_1^n$ (uniformly over n) and—following Szarek's proof [S]—this holds more generally if X has uniformly bounded volume ratio, i.e. if

$$\sup\{vr(E) \mid E \subset X \ \ \dim E < \infty\} < \infty. \tag{0.9}$$

The question was then raised whether these two different approaches are related and specifically whether cotype 2 implies (0.9). This was proved by Bourgain and Milman in [BM]. Shortly after, this was continued (with somewhat different methods) by Milman and the author who showed in particular that (*) implies (0.9) (cf. [MP1]). In the same paper, a weakened version of the notion of cotype 2 (*weak cotype 2*) was introduced and was shown to be equivalent to (*) and (0.9).

Chapter 10 is mainly devoted to these results of [MP1].

In Chapter 11, we present the notion of *weak type 2* which is somewhat dual to the preceding. A space X is weak type 2 iff its dual X^* is K-convex and weak cotype 2, cf. [MP1]. We also include a characterization of weak type 2 in terms of volume ratio due to A. Pajor [Pa1].

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Gilles Pisier
Frontmatter
More information

In Chapter 12 we discuss the *weak Hilbert spaces*, i.e. the spaces which are both weak cotype 2 and weak type 2. This chapter is based on [P5]. For instance, we show that X is a weak Hilbert space iff one of the following properties (a), (b) holds.

(a) *There is a constant C such that for all f.d. subspaces $E \subset X$ there are ellipsoids D_1, D_2 in E such that*

$$D_1 \subset B_X \cap E \subset D_2$$

and

$$\left(\frac{\mathrm{vol}(D_2)}{\mathrm{vol}(D_1)}\right)^{1/n} \leq C.$$

(b) *There is a constant C such that for all n and all $x_1, \ldots, x_n$, $x_1^*, \ldots, x_n^*$ in the unit balls respectively of X and X^* we have*

$$\left|\det\left(< x_i^*, x_j >\right)\right|^{1/n} \leq C.$$

In Chapter 13 we present some examples of weak Hilbert spaces which are not isomorphic to Hilbert spaces. These examples are due to W. B. Johnson (see [FLM]) but are based on an earlier example of Tsirelson of a reflexive Banach space which contains ℓ_p for no p (cf. [T], [FJ2]). Since such spaces have an unconditional basis, it would be unreasonable to dismiss them as *pathological.* We refer the reader to the forthcoming book of Casazza and Shura [CS] for more information and similar examples.

In Chapter 14 we present some ideas due to W. B. Johnson which show that weak Hilbert spaces are reflexive, and hence super-reflexive.

In Chapter 15 we use the theory of determinants (sometimes called the Fredholm Theory) for elements of the projective tensor product $X^* \widehat{\otimes} X$ as described in [G2] to show that weak Hilbert spaces have the Approximation Property, and hence the Uniform Approximation Property. We also prove that the so-called Lidskii trace formula remains valid in a weak Hilbert space X: Let T be a nuclear operator on X; then, if its eigenvalues $\{\lambda_n\}$ are absolutely summable, we have

$$tr(T) = \sum \lambda_n.$$

In general, however, unless X is isomorphic to a Hilbert space, the eigenvalues of T are not absolutely summable (cf. [JKMR]).

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Gilles Pisier
Frontmatter
More information

If X is weak Hilbert, and if the eigenvalues are rearranged so that $|\lambda_1| \geq |\lambda_2| \geq, \ldots$, then any nuclear operator satisfies $\sup_n n|\lambda_n| < \infty$. In other words, instead of (λ_n) in ℓ_1 we find (λ_n) in *weak*-ℓ_1 (the space $\ell_{1\infty}$) and this characterizes weak Hilbert spaces. Analogous statements hold for weak cotype 2 and weak type 2 (see [P5] for a broader notion of weak-P when P is a given property of Banach spaces). This explains in part the choice of the adjective *weak* common to all these notions.

We should warn the reader, however, that the space often called *weak*-L_2 (often denoted $L_{2\infty}$) is not a weak Hilbert space; it is well known that this space contains an isomorphic copy of ℓ_∞.

In general, we do not give references in the text, but in the **Notes and Remarks** following each chapter. There are a number of exceptions to this rule, but, in any case, we warn the reader *always* to consult the **Notes and Remarks** section to find out to whom a given result should be credited.

We apologize in advance for possible errors or for references which may have been omitted for lack of accurate information.

This book is based on a series of lectures given at the University of Paris VI (spring 1985) and on graduate courses given in Paris VI (spring 1986) and at Texas A&M University (fall 1986). I am grateful to all those who participated in these lectures and especially to Alvaro Arias, who also did a careful proofreading of the first draft. I am also grateful to N. W. Naugle for his expert advice on TEX and for his help in dealing with diagrams. Finally, special thanks are due to Jan Want for her excellent typing.

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