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# Chapter 1

# Notation and Preliminary Background

Let $E, F$ be Banach spaces. If $E, F$ are isomorphic, we define their *Banach–Mazur distance* as

$$d(E,F) = \inf\{\|T\|\ \|T^{-1}\|\} \tag{1.1}$$

where the infimum runs all isomorphisms $T : E \to F$. If they are not isomorphic, we set $d(E,F) = +\infty$. If $E, F$ are isomorphic and if $d(E,F) \leq \lambda$, we will say briefly that $E$ and $F$ are $\lambda$-isomorphic (or that $E$ is $\lambda$-isomorphic to $F$).

For all Banach spaces, $E, F, G$ we have obviously

$$d(E,G) \leq d(E,F)d(F,G).$$

We denote by $B_E$ the unit ball of $E$ and by $I_E$ the identity operator on $E$.

In most of these notes we work with *finite dimensional* Banach spaces $E, F$. We will abreviate finite dimensional by f.d. The above *distance* is then particularly useful since any two spaces with the same dimension are isomorphic. Let $K(n)$ be the set of all normed (or Banach) spaces of dimension $n$. For $E, F$ in $K(n)$, a simple compactness argument shows that the infimum is attained in (1.1) so that $E, F$ are *isometric* iff $d(E,F) = 1$. The relation $E$ *is isometric to* $F$ is clearly an equivalence relation on $K(n)$. Let us denote by $\widetilde{K}(n)$ the set of all classes modulo this equivalence. Then it is not hard to check that $\widetilde{K}(n)$ equipped with the metric $\delta(E,F) = \operatorname{Log} d(E,F)$ is a compact metric space, sometimes called the Banach–Mazur compactum.

We will denote by $\ell_p^n$ the space $\mathbf{R}^n$ equipped with the norm $\|x\| = (\sum_1^n |x_i|^p)^{1/p}$. In particular, $\ell_2^n$ is the $n$-dimensional Euclidean space. The latter space plays *a central role* among the elements of $K(n)$ (or of $\widetilde{K}(n)$).

In particular, we show in Chapter 3 a classical result of F. John [Joh]

$$d(E, \ell_2^n) \leq n^{1/2} \quad \text{for all} \quad E \quad \text{in } K(n).$$

There has been in recent years a great deal of progress concerning the *local theory* of Banach spaces. This is the part of Banach space theory which uses mainly finite-dimensional tools and methods. In this theory an infinite-dimensional space is studied through the collection of all its finite-dimensional subspaces. For instance, by a fundamental theorem of Dvoretzky (cf. Chapter 4), every-infinite dimensional space $X$ contains for each $n$ and $\varepsilon > 0$ a subspace $E$ such that $d(E, \ell_2^n) < 1 + \varepsilon$. Recently these methods have been successfully applied to prove several inequalities on the volume of convex symmetric bodies in $\mathbf{R}^n$. We will call these simply *balls*. More precisely, throughout the sequel, a *ball* will be a compact convex symmetric subset $B \subset \mathbf{R}^n$ with non-empty interior. Let $\| \ \|_B$ be the gauge of $B$. For any ball $B \subset \mathbf{R}^n$, the space $\mathbf{R}^n$ equipped with $\| \ \|_B$ is a Banach space admitting $B$ as its unit ball. Conversely, any $n$-dimensional normed space $E$ can be identified (in more than one way) with $\mathbf{R}^n$ and hence we may associate to $E$ its unit ball $B_E \subset \mathbf{R}^n$. Note that two balls $B_1, B_2$ in $\mathbf{R}^n$ correspond to two isometric Banach spaces iff there is a linear isomorphism $u : \mathbf{R}^n \to \mathbf{R}^n$ such that $u(B_1) = B_2$.

If the normed space associated to a ball $B \subset \mathbf{R}^n$ is a Hilbert space, then $B$ is an ellipsoid, i.e. there is an isomorphism $u : \mathbf{R}^n \to \mathbf{R}^n$ which maps $B$ onto the canonical Euclidean ball $B_{\ell_2^n}$.

It will be useful to recognize geometrically the balls of subspaces and quotient spaces of an $n$-dimensional normed space $E$ with unit ball $B_E \subset \mathbf{R}^n$. For subspaces this is immediate: let $F$ be a subspace of $E$; clearly the *section* $F \cap B_E$ can be viewed as the unit ball of the normed subspace $F$. We may also consider the quotient space $E/F$. Geometrically, this space corresponds not to sections of $B_E$, but to linear projections of $B_E$. Indeed, let $P : E \to E$ be any linear projection such that $\ker P = F$. Let $G$ be the range of $P$. We equip $G$ with the norm which admits $P(B_E)$ as its unit ball. Then $G$ is isometric to $E/F$. In particular, we may wish to use the orthogonal projection $P_G$ into $G$, orthogonal with respect to a fixed scalar product on $\mathbf{R}^n$ (for instance the usual one). Then $P_G(B_E)$ can be naturally identified with the unit ball of the normed space $E/G^\perp$.

In the sequel, we always denote by $E^*$ the dual space of $E$ equipped with the dual norm. Recall that for $K \subset \mathbf{R}^n$, the polar set is defined as

$$K^\circ = \{x \in \mathbf{R}^n | \ < x, y > \le 1 \quad \forall\, y \in K\}.$$

Clearly, if $B_E \subset \mathbf{R}^n$ is the unit ball of $E$, we may identify $B_{E^*}$ with the polar $(B_E)^\circ$ of $B_E$. We note the obvious identities for all subspaces $F \subset \mathbf{R}^n$:

$$(F \cap B_E)^\circ = P_F(B_E^\circ) \qquad (P_F(B_E))^\circ = F \cap (B_E^\circ).$$

(Here the first and third polar sets are with respect to $F$ with the induced scalar product.)

We now state and prove the Brunn–Minkowski inequality. Although we do not really need this inequality (or its corollaries) in these notes, it is natural to include it here.

**Theorem 1.1.** *Let $A, B$ be two compact subsets of $\mathbf{R}^n$, then for all $\lambda$ in [0,1] we have*

$$\text{(1.1)} \qquad \operatorname{vol}(\lambda A + (1-\lambda)B) \geq \operatorname{vol}(A)^\lambda \operatorname{vol}(B)^{1-\lambda},$$

*and*

$$\text{(BM)} \qquad (\operatorname{vol}(A+B))^{1/n} \geq (\operatorname{vol}(A))^{1/n} + (\operatorname{vol}(B))^{1/n}.$$

We have learned the proof below from Keith Ball (cf. [Ba1]). It is based on ideas from the papers [Pr] and [Le]. (cf. also [BrL]). We first prove a lemma.

**Lemma 1.2.** *Let $f, g, \phi$ be measurable functions from $\mathbf{R}^n$ into $[0,\infty]$ such that for some $0 < \lambda < 1$ and all $r, s$ in $\mathbf{R}^n$ we have*

$$\phi(\lambda r + (1-\lambda)s) \geq f(r)^\lambda g(s)^{1-\lambda}.$$

*Then, denoting by $m$ the Lebesgue measure on $\mathbf{R}^n$, we have*

$$\text{(1.2)} \qquad \int \phi\, dm \geq \left(\int f\ dm\right)^\lambda \left(\int g\ dm\right)^{1-\lambda}.$$

**Proof:** We first consider the case $n = 1$. We may clearly assume $f, g$ bounded and (by homogeneity) satisfying $\|f\|_\infty = \|g\|_\infty = 1$.

Note that for any $0 \leq a < 1$, we have obviously

$$\{\phi \geq a\} \supset \lambda\{f \geq a\} + (1-\lambda)\{g \geq a\}$$

and since both sets on the right are non-empty, this implies (by the one-dimensional Brunn–Minkowski inequality!)

$$m\{\phi \geq a\} \geq \lambda\, m\{f \geq a\} + (1-\lambda) m\{g \geq a\}.$$

After integration in $a$ over $[0, \infty]$, this implies

$$\int \phi\ dm = \int \{\phi \geq a\}\, dm \geq \lambda \int f\ dm + (1-\lambda) \int g\ dm;$$

hence (by the arithmetic–geometric mean inequality)

$$\geq \left(\int f\ dm\right)^{\lambda} \left(\int g\ dm\right)^{1-\lambda}.$$

This completes the proof of (1.2) for $n = 1$.

It is then easy to deduce the case of $\mathbf{R}^n$ by induction on $n$. Suppose $n > 1$ and assume the lemma proved for $n-1$. Consider $\phi, f, g$ from $\mathbf{R}^n$ into $[0, \infty]$. Let $y \in \mathbf{R}$ be fixed, define $\phi_y : \mathbf{R}^{n-1} \to [0, \infty]$ by $\phi_y(t) = \phi(t, y)$ and similarly for $f_y$ and $g_y$. Clearly, we have, if

$$y = \lambda y_1 + (1-\lambda) y_0 \quad (y_0, y_1 \in \mathbf{R}), \quad \phi_y(\lambda r + (1-\lambda) s) \geq f_{y_1}(r)^{\lambda} g_{y_0}(s)^{1-\lambda}$$

for any $r, s$ in $\mathbf{R}^{n-1}$. Therefore by the induction hypothesis,

$$\int_{\mathbf{R}^{n-1}} \phi_y \geq \left(\int_{\mathbf{R}^{n-1}} f_{y_1}\right)^{\lambda} \left(\int_{\mathbf{R}^{n-1}} g_{y_0}\right)^{1-\lambda}.$$

Finally, applying (1.2) one more time (here for $n = 1$), we obtain

$$\int \phi\ dm = \int \left(\int_{\mathbf{R}^{n-1}} \phi_y\right) dy \geq \left(\int f\ dm\right)^{\lambda} \left(\int g\ dm\right)^{1-\lambda}.$$

This completes the proof of Lemma 1.2. ■

**Proof of Theorem 1.1:** The inequality (1.1) follows immediately from (1.2) with

$$\phi = 1_{\lambda A + (1-\lambda) B}, \quad f = 1_A, \quad g = 1_B.$$

Then the Brunn–Minkowski inequality (BM) follows by setting

$$\lambda = (\mathrm{vol}(A))^{1/n} (\mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n})^{-1}.$$

Note that if $A' = \mathrm{vol}(A)^{-1/n}A$ and $B' = \mathrm{vol}(B)^{-1/n}B$, then (1.1) implies

$$\mathrm{vol}(\lambda A' + (1-\lambda)B') \geq 1, \tag{1.3}$$

and, since

$$\lambda A' + (1-\lambda)B' = \frac{A+B}{(\mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n})},$$

(1.3) immediately implies (BM) by homogeneity. ■

**Remark:** For a different proof of (BM) and more information, we refer the interested reader to [Be], [BZ], [Eg] [BF], [Ha1] and [Ha2]. The reader who wishes to learn about recent developments in the Classical Theory of Convex Sets should consult for instance [Sa2] and the collection of surveys [GW].

As a corollary, we can prove the classical isoperimetric inequality in $\mathbf{R}^n$. We recall that the area of the boundary of a convex compact subset of $\mathbf{R}^n$ can be defined by an approximating procedure from the simpler case of polytopes (cf. e.g. [Be] or [Eg].)

**Corollary 1.3.** (Isoperimetric inequality) *Let $C$ be a compact convex subset of $\mathbf{R}^n$ with non-empty interior. Let us denote by $a(C)$ the area of its boundary. Let $B_2$ be the canonical euclidean ball in $\mathbf{R}^n$. Then*

$$\left(\frac{\mathrm{vol}(C)}{\mathrm{vol}(B_2)}\right)^{1/n} \leq \left(\frac{a(C)}{a(B_2)}\right)^{1/n-1}.$$

**Proof:** Indeed, the area can be *derived* from the volume in a simple way, since we have

$$a(C) = \lim_{t\to 0} t^{-1}(\mathrm{vol}(C+tB_2) - \mathrm{vol}(C)). \tag{1.4}$$

By (1.4), we have $a(B_2) = n\ \mathrm{vol}(B_2)$ and by (BM)

$$\begin{aligned}\mathrm{vol}(C+tB_2) &\geq (\mathrm{vol}(C)^{1/n} + t\ \mathrm{vol}(B_2)^{1/n})^n \\ &\geq \mathrm{vol}(C) + nt\ \mathrm{vol}(B_2)^{1/n}\,\mathrm{vol}(C)^{\frac{n-1}{n}} + o(t)\end{aligned};$$

hence

$$a(C) \geq \left(\frac{\mathrm{vol}(C)}{\mathrm{vol}(B_2)}\right)^{\frac{n-1}{n}} \cdot a(B_2). \quad ■$$

We can also deduce from (BM) a classical inequality of Urysohn [U] which we do not really use in the sequel, but which clarifies the relationship between our methods and classical inequalities such as (BM). We denote by $S$ the Euclidean unit sphere of $\mathbf{R}^n$ and by $\sigma$ its normalized area measure.

**Corollary 1.4.** (Urysohn's Inequality) *Let $K$ be a compact subset of $\mathbf{R}^n$. Let*

$$\|x\|_{K^\circ} = \sup\{< x, y >, y \in K\}$$

*for all $x \in \mathbf{R}^n$. Then*

$$\left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B_2)}\right)^{1/n} \leq \int_S \|x\|_{K^\circ}\, d\sigma(x).$$

**Proof:** It is easy to generalize (BM) to a finite number of sets $A_1, \dots,$ $A_k$ so that for any numbers $m_i \geq 0$ we have

$$\sum m_i \,\mathrm{vol}(A_i)^{1/n} \leq \left(\mathrm{vol}\left(\sum m_i A_i\right)\right)^{1/n}.$$

More generally (under suitable assumptions), if $A_t$ depends on a parameter $t$ varying in a measure space $(\Omega, \sum, \nu)$ then we have

$$\text{(1.5)} \qquad \int \mathrm{vol}(A_t)^{1/n}\, d\nu(t) \leq \left(\mathrm{vol}\left(\int A_t\, d\nu(t)\right)\right)^{1/n}.$$

Now let $(\Omega, \nu)$ be the orthogonal group $O(n)$ equipped with its normalized Haar measure. Let $A_t = t(K)$ for all $t$ in $O(n)$. Then $\mathrm{vol}(A_t) = \mathrm{vol}(K)$ for all $t$. On the other hand, $\int t(K)\, d\nu(t)$ is clearly (by symmetry) a multiple of the Euclidean ball $B_2$. Hence

$$\text{(1.6)} \qquad \int t(K)\, d\nu(t) = \lambda B_2$$

for some number $\lambda \geq 0$.

Now let $\xi$ be a fixed element of $S$. (For instance, let $\xi = e_1$ the first basis vector of $\mathbf{R}^n$.) Writing that the images under $\xi$ of both sides of (1.6) coincide, we find

$$\int \xi(t(K))\, d\nu(t) = \lambda[-1, 1].$$

Clearly, $\xi(t(K)) = [a_t, b_t]$ with $a_t = \inf_{x \in K} \xi(t(x))$ and $b_t = \sup_{x \in K} \xi(t(x))$. This implies

$$\int b_t\, d\nu(t) = \lambda,$$

so that (1.5) implies

$$\mathrm{vol}(K)^{1/n} \leq \lambda(\mathrm{vol}(B_2))^{1/n}.$$

This completes the proof since $\lambda = \int \|t^*\xi\|_{K^\circ} d\nu(t) = \int \|x\|_{K^\circ} d\sigma(x)$. ■

The preceding proof was shown to me by V. Milman.

**Remark 1.5:** Let $\gamma$ be the canonical Gaussian probability measure on $\mathbf{R}^n$. Then, integrating in polar coordinates, we find

$$\int \|x\|_{K^\circ}\, d\gamma(x) = c_n \int_S \|x\|_{K^\circ}\, d\sigma(x),$$

with $c_n = \int (\sum_1^n x_i^2)^{1/2}\, d\gamma(x) \le \sqrt{n}$ and $\frac{c_n}{\sqrt{n}} \to 1$ when $n \to \infty$. Thus, Corollary 1.5 implies *a fortiori*

$$\left(\frac{\text{vol}(K)}{\text{vol}(B_2)}\right)^{1/n} \le n^{-1/2} \left(\int \|x\|_{K^\circ}^2\, d\gamma(x)\right)^{1/2}.$$

This last inequality explains typically why the estimates of Gaussian integrals of the form $\int \|x\|^2\, d\gamma(x)$ (for some norm $\|\ \|$ on $\mathbf{R}^n$) can be used to evaluate certain volumes. In the present notes, we will exploit the recent techniques of the local theory of Banach spaces concerning integrals such as $\int \|x\|^2 d\gamma(x)$ to obtain some new inequalities on the volumes of convex bodies in $\mathbf{R}^n$.

In the second part of this chapter, we recall some basic facts of operator theory. An operator $T : H_1 \to H_2$ between Hilbert spaces is called a Hilbert–Schmidt operator if for some (equivalently for all) orthonormal basis $(e_i)$ of $H_1$ we have $\sum \|T\ e_i\|^2 < \infty$. The *Hilbert–Schmidt norm* of $T$ is then defined as

$$\|T\|_{HS} = \left(\sum \|T\ e_i\|^2\right)^{1/2}.$$

Let $(f_j)$ be an orthonormal basis in $H_2$. We have

$$(1.7) \qquad \|T\|_{HS}^2 = \sum_{ij} \left| < Te_i, f_j > \right|^2 = \sum_j \|T^* f_j\|^2 = \|T^*\|_{HS}^2.$$

This shows that $\|T\|_{HS}$ is independent of the choice of the basis $(e_i)$ and we have

$$(1.8) \qquad \|T\|_{HS} = \|T^*\|_{HS}.$$

We also note the identity

$$(1.9) \qquad \begin{aligned} \|T\|_{HS}^2 &= \sum < Te_i, Te_i > = \sum < T^*Te_i, e_i > \\ &= tr\ T^*T = tr\ |T|^2 \end{aligned}$$

which shows that if $T$ is Hilbert–Schmidt, then $T^*T = |T|^2$ is a trace class operator.

Recall that a Hilbert–Schmidt operator is *a fortiori* compact. For any compact operator $T$, we will denote by $(\lambda_i(T))_{i\geq 1}$ the eigenvalues of $T$ repeated according to their multiplicities and arranged so that the sequence $\{|\lambda_n(T)|, n \geq 0\}$ is non-increasing. Let $T = U|T|$ be the polar decomposition of $T$ where $|T| = (T^*T)^{1/2}$ is a positive self-adjoint operator and $U$ is a partial isometry (it is isometric on the range of $|T|$).

Then clearly $\|T\|_{HS} < \infty$ iff $\| \, |T| \, \|_{HS} < \infty$, and we have

$$\|T\|_{HS} = \| \, |T| \, \|_{HS}. \tag{1.10}$$

From (1.9) we derive

$$\|T\|_{HS} = \left(\sum_1^\infty \lambda_n(|T|)^2\right)^{1/2}. \tag{1.11}$$

The numbers $\lambda_n(|T|)$ can also be characterized as the approximation numbers of $T$. Let $T : X \to Y$ be an operator between Banach spaces. We denote as usual

$$a_n(T) = \inf\{\|T - S\| \mid S : X \to Y \; rk(S) < n\}.$$

Note that $\|T\| = a_1(T) \geq a_2(T) \geq \dots$ . In the particular case of an operator $T : H_1 \to H_2$ between Hilbert spaces, it is well known that

$$a_n(T) = \lambda_n(|T|), \tag{1.12}$$

so that

$$\|T\|_{HS} = \left(\sum a_n(T)^2\right)^{1/2}. \tag{1.13}$$

In the Banach space case, the class of *p-summing operators* introduced by Pietsch following the work of Grothendieck (cf. [Pi1]) replaces quite often (especially when $p = 2$) the class of Hilbert–Schmidt operators. Let $0 < p < \infty$. Recall that an operator $T : X \to Y$ between Banach spaces is called $p$-summing if there is a constant $C$ such that for all finite sequences $(x_i)$ in $X$, we have

$$\left(\sum \|Tx_i\|^p\right)^{1/p} \leq C \sup\left\{\left(\sum \left|\xi(x_i)\right|^p\right)^{1/p} \middle| \xi \in B_{X^*}\right\}.$$

We denote by $\pi_p(T)$ the smallest constant $C$ for which this holds and by $\Pi_p(X,Y)$ the set of all $p$-summing operators from $X$ into $Y$. This set is a Banach space when equipped with the norm $\pi_p$. These operators form an *operator ideal* in the sense of Pietsch, which essentially means that for all Banach spaces $X_1, Y_1$ and all operators $u : X_1 \to X$ and $v : Y \to Y_1$, if $T : X \to Y$ is $p$-summing, then $vTu$ is also and we have

$$\pi_p(vTu) \le \|v\|\pi_p(T)\|u\|. \tag{1.14}$$

This is easy to check from the definition. Moreover, if $T : X \to Y$ is $p$-summing and if $\phi$ is in $L_p(\Omega,\mu;X)$ (here $(\Omega,\mu)$ denotes an arbitrary measure space) we have

$$\|T(\phi)\|_{L_p(Y)} \le \pi_p(T)\sup\{\|\xi(\phi)\|_p \mid \xi \in B_{X^*}\}. \tag{1.15}$$

Indeed, this is easy to check for step functions and it can be extended to $L_p(X)$ by density. The space $L_p(\Omega,\mu;X)$ always denotes in the sequel the completion of $L_p(\Omega,\mu)\otimes X$ with the usual norm.

We will use the following well-known fact:

**Proposition 1.6.** *Let $T : H_1 \to H_2$ be an operator between two Hilbert spaces. Then $T$ is 2-summing iff $T$ is Hilbert–Schmidt and we have*

$$\pi_2(T) = \|T\|_{HS}.$$

**Proof:** We will prove the *only if* part first. So consider a 2-summing operator $T : H_1 \to H_2$ and let $(e_i)$ be an orthonormal basis of $H_1$. We have

$$\begin{aligned}\left(\sum \|T e_i\|^2\right)^{1/2} &\le \pi_2(T)\sup\left\{\left(\sum|\xi(e_i)|^2\right)^{1/2} \Big| \|\xi\| \le 1\right\}\\ &\le \pi_2(T).\end{aligned}$$

Hence, $T$ is Hilbert–Schmidt and $\|T\|_{HS} \le \pi_2(T)$.

Conversely, assume that $T$ is Hilbert–Schmidt. Let $x_1,\dots,x_n$ be arbitrary in $H_1$ and let $(f_j)$ be an orthonormal basis of $H_2$. We have

$$\begin{aligned}\sum \|Tx_i\|^2 &= \sum_{ij} |<f_j, Tx_i>|^2\\ &= \sum_j\left(\sum_i |<T^* f_j, x_i>|^2\right)\\ &\le \sum_j \|T^* f_j\|^2 \sup\left\{\sum |\xi(x_i)|^2 \Big| \|\xi\| \le 1\right\}.\end{aligned}$$

This shows that $T$ is 2-summing and by (1.7) we have $\pi_2(T) \le \|T\|_{HS}$. ■

As a corollary, we have

**Corollary 1.7.** *Let $E_1, E_2$ be Banach spaces which are isomophic respectively to the Hilbert spaces $H_1$ and $H_2$. Then every 2-summing operator $T : E_1 \to E_2$ satisfies the expression*

$$\left(\sum a_n(T)^2\right)^{1/2} \le d(E_1, H_1) d(E_2, H_2) \pi_2(T). \tag{1.16}$$

**Proof:** This is immediate using (1.14) and (1.13) together with Proposition 1.6.

For various purposes, we will also need the following simple fact first used in a similar context by D. Lewis.

**Lemma 1.8.** *Let $T : H \to X$ be an operator from a Hilbert space into a Banach space. For every $\varepsilon > 0$, there is an orthonormal system $(f_n)$ in $H$ such that $\|T f_n\| \ge a_n(T) - \varepsilon$ for all $n \ge 1$. Moreover, if $H$ is f.d., we have this also for $\varepsilon = 0$.*

**Proof:** Let $f_1 \in H$ be such that $\|f_1\| = 1$ and

$$\|T(f_1)\| \ge \|T\| - \varepsilon = a_1(T) - \varepsilon.$$

We consider the restriction of $T$ to $S_1 = [f_1]^\perp$ and note that obviously $\|T_{|S_1}\| \ge a_2(T)$. Therefore, there is a norm 1 element $f_2 \in [f_1]^\perp$ such that

$$\|T(f_2)\| \ge a_2(T) - \varepsilon.$$

We then restrict $T$ to $[f_1, f_2]^\perp$. Continuing in this way, we obtain a sequence $(f_n)$ which has the announced property. In the f.d. case, since the unit balls are compact, we may take $\varepsilon = 0$. ■

With this result we can improve Corollary 1.7 as follows.

**Corollary 1.9.** *Let $u : H \to X$ be a 2-summing operator from a Hilbert space into a Banach space. Then $(\sum a_k(u)^2)^{1/2} \le \pi_2(u)$.*

**Proof:** Let $\varepsilon > 0$. Let $(f_k)$ be an orthonormal basis such that $\|u f_k\| \ge a_k(u) - \varepsilon$. Then

$$\left(\sum \|u f_k\|^2\right)^{1/2} \le \pi_2(u) \sup\left\{\left(\sum |<\xi, f_k>|^2\right)^{1/2} \|\xi\| \le 1\right\}.$$