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Spectral Asymptotics in the Semi-Classical Limit

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0. Introduction

A new branch of mathematical analysis, so-called microlocal analysis, started to be more systematically developed about 30 years ago by Kohn–Nirenberg, Hörmander, Maslov and Sato, soon followed by many others. Originally the motivations came from problems in partial differential equations, but it soon became increasingly clear that many aspects of microlocal analysis are reminiscent of quantum mechanics, and for instance the Heisenberg uncertainty principle plays a fundamental role in both theories. Mathematically, a version of this principle says that if $u \in L^2(\mathbf{R}^n)$ and we define the Fourier transform by

$$\widehat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)dx, \tag{0.1}$$

so that Parseval's relation

$$\|u\|^2 = \frac{1}{(2\pi)^n}\|\widehat{u}\|^2,$$

holds, where the norms are those of L^2, then if we take $n = 1$ for simplicity, and let $x_0, \xi_0 \in \mathbf{R}$:

$$\|u\|^2 \le 2\|(\cdot - x_0)u\| \frac{1}{\sqrt{2\pi}} \|(\cdot - \xi_0)\widehat{u}\|, \quad u \in \mathcal{S}(\mathbf{R}). \tag{0.2}$$

Here $\mathcal{S}(\mathbf{R})$ is the Schwartz space of smooth functions on $\mathbf{R}$ which decay rapidly at infinity together with all their derivatives. A rough interpretation of this is that if most of the L^2-norm ('energy') of u is concentrated to an interval of length a and most of the energy of $\widehat{u}$ is concentrated to an interval of length b, then:

$$ab \ge 2\pi. \tag{0.3}$$

The reason for putting this precise numerical constant comes from well-known asymptotic formulas for the counting of eigenvalues (of Weyl type) which can be interpreted by saying that each eigenfunction occupies a volume (2π) in phase space.

Another similarity between the two theories is the interplay between classical and quantum objects. In microlocal analysis, the quantum objects are given by pseudodifferential and Fourier integral operators etc. and the classical ones by those of symplectic geometry: canonical transformations, Poisson brackets etc. In quantum mechanics the same duality appears in the semi-classical limit. If we consider for instance the stationary Schrödinger operator

$$-h^2\Delta + V(x), \tag{0.4}$$

when h becomes very small, then the quantum objects are wave functions, eigenvalues etc., while the classical ones are given by the classical trajectories of the associated classical Hamiltonian $p := \xi^2 + V(x)$, i.e. the integral curves of the corresponding Hamilton field $H_p = 2\xi \cdot (\partial/\partial x) - V'(x) \cdot (\partial/\partial \xi)$.

Thanks to microlocal analysis it has been possible to get refined results about the distributions of eigenvalues for differential operators (mostly elliptic ones) on compact manifolds and in bounded domains (Hörmander, Duistermaat–Guillemin, Ivrii and others), and while the Weyl asymptotics gives the leading terms in such results and is simply a phase space volume, the further terms or remainder estimates depend on dynamical properties of the Hamilton flow.

These notes are about the analogous developments for the semiclassical limit. The motivation among specialists (such as Chazarain, Helffer–Robert and later many others) was that microlocal analysis should provide a tool for a more rigorous understanding of many spectral problems also in this field. To some extent the early work consisted of carrying over the above mentioned spectral results to the study of, say, (0.4), but the area turned out to be much richer and new problems and results appeared, and the microlocal analysis itself has received new impulses from these efforts.

The contents of these notes are:

1. Local symplectic geometry. Here we develop some of the standard theory, following closely one of the chapters in [GrSj].

2. The WKB-method. We discuss the construction of local asymptotic solutions of $(P - E)u = 0$, where P is the operator (0.4), and get an example of the interplay between classical and quantum objects.

3. The WKB-method for a potential minimum. Here we follow some work by Helffer and one of the authors, and show how to construct asymptotic eigenvalues and eigenfunctions near a non-degenerate minimum of the potential.

4. Self-adjoint operators. This is mostly a compilation of abstract spectral theory, and at the end of the chapter, we determine the low-lying eigenvalues for potentials with a non-degenerate minimum. This also justifies the more complete asymptotics of eigenvalues obtained in Chapter 3.

5. The method of stationary phase. We followed closely the presentation of [GrSj], based on the classical work of Hörmander [Hö1]. A small variation leads to some refined remainder estimates, which may be new. This method is one of the fundamental ingredients of microlocal analysis, even though in the present notes we choose not to appeal explicitly to this method when presenting the theory of pseudodifferential operators. ([GrSj] shows how to get everything from stationary phase.)

6. Tunnel effect and interaction matrix. This chapter is devoted to exponentially small corrections to eigenvalues of (0.4), due to the interaction of potential wells through the classically forbidden region. An essential tool is the use of exponentially weighted L^2-estimates, developed for second order operators by Lithner and Agmon. We have followed some work of Helffer and one of the authors.

7. h-pseudodifferential operators. In this chapter the basic theory of pseudodifferential operators is developed, without trying to reach maximal generality or refinement. These operators are of the form $P(x, hD; h)$, where $P(x, \xi; h)$ belongs to some suitable space of symbols. The most standard case is when $P(x, \xi; h)$ is uniformly bounded together with all its derivatives, uniformly with respect to h. In the case $n = 1$ (for simplicity) the symbol $P(x, h\xi; h)$ varies only a little in rectangles of the form $I_x \times J_\xi$ if I_x and J_ξ are intervals of length ϵ_0 and ϵ_0/h respectively, for some small but fixed constant $\epsilon_0 > 0$. The area of $I_x \times J_\xi$ is ϵ_0^2/h, and the uncertainty principle is satisfied with a good margin, when h is small enough. The symbolic calculus is developed and in particular it is established that h-pseudodifferential operators form an algebra, and the symbols of the composition of two operators is the product of the symbols plus an error which is roughly of the order h smaller.

8. Functional calculus for pseudodifferential operators. We base this calculus on a functional formula using almost analytic extensions, and a semi-classical version of an important lemma of Beals which permits us to characterize pseudodifferential operators. One of the main results (which is due to Helffer and Robert in the semi-classical case) says that if P is a self-adjoint h-pseudodifferential operator (from now on sometimes called pseudor for short) and $f \in C_0^\infty$, then $f(P)$ is again a h-pseudor with leading symbol $f(p(x, \xi))$, where $p(x, \xi)$ is the leading symbol of P. We follow some joint papers of Helffer and one of the authors. This approach to the calculus is also extended to the case of several commuting self-adjoint operators.

9. Trace class operators and applications of the functional calculus. Here we derive asymptotic expansions for the trace and get as a corollary the leading (Weyl-)asymptotics for the number of eigenvalues in an interval.

10. More precise spectral asymptotics for non-critical Hamiltonians. Here we study the unitary evolution group as a Fourier integral operator and the singularity of its trace near the time 0. This leads to an estimate of the remainder in the spectral asymptotic formula, which in general is optimal.

11. Improvement when the periodic trajectories form a set of measure 0. Here we estimate the trace of the evolution group also for large times. The methods and the results of this chapter as well as the preceding one are fairly standard, first due to Hörmander, Guillemin–Duistermaat in the non-semiclassical case, then extended to the semi-classical case (and improved) by Ivrii, Petkov and Robert.

12. A more general study of the trace. Here we extend the results of Chapter 10 to the case of microhyperbolic systems. The presentation is inspired by works of Ivrii, which avoid explicit constructions (which might be impossible anyway), but we have used a stationary approach, which in later work by one of the authors has been extended to situations with an implicit dependence of the spectral parameter. Such implicit spectral problems appear frequently when making so-called Grushin reductions of a spectral problem.

13. Spectral theory for perturbed periodic problems. For slowly varying perturbations of periodic Schrödinger operators, one can make a reduction to the study of an h-pseudor, a so called effective Hamiltonian, and it then becomes possible to obtain asymptotic results about the eigenvalues of the perturbed operator in a gap of the spectrum of an unperturbed one. We have followed work by Gérard–Martinez–Sjöstrand and Dimassi, related to earlier works by Buslaev, Guillot–Ralston–Trubowitz and Helffer–Sjöstrand. The reduction used is an example of a so-called Grushin reduction, a technique which has turned out to be extremely useful in many situations, in particular when combined with functional formulas of the type given in Chapters 8, 9 and 12.

14. Normal forms for some scalar pseudodifferential operators. Here we return to non-degenerate potential wells, studied in Chapters 3, 4 and 6, and establish a quantum Birkhoff normal form, which permits (under a non-resonance condition) to obtain complete asymptotic expansions of all eigenvalues in an interval $[0, h^{\delta}]$, where $\delta > 0$ is arbitrary. This chapter is based on a work of Sjöstrand, in a cirle of ideas developed by Lazutkin, Colin de Verdière, Graffi–Paul, Bellissard–Vittot, Iantchenko and many others.

15. Spectrum of operators with periodic bicharacteristics. When the Hamilton flow is periodic there is a phenomenon of clustering of eigenvalues, that we study, following works by Colin de Verdière, Weinstein, Helffer–Robert and others.

We hope that these notes may serve as an introduction to a still very active subject, and they correspond largely to a course given by the authors at the universities of Rennes (,where the first impulse to write these notes was received), Paris Sud, Paris Nord, as well as the Ecole Polytechnique. They cover more recent material than the now classical book by Robert [Ro1], but remain hopefully at an introductory level. A fairly large portion can be covered in a one semester course. For further and deeper study, we can recommend the recent book by Ivrii [I1]. See also the book of

Safarov–Vassiliev [SaVa] which deals with asymptotics of large eigenvalues for boundary value problems.

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