1. Local symplectic geometry

We assume that the reader is familiar with some basic objects of differential geometry, such as manifolds, tangent and cotangent vectors, differential forms and vector bundles. We shall, however, review briefly some of these notions. For a smooth manifold X of dimension n we shall denote by $C^k(X)$ the space of k times continuously differentiable complex valued functions on X if $k \in \mathbf{N}$, and we set $C^\infty(X) = \cap_{k\in\mathbf{N}} C^k(X)$.

Tangent and cotangent vectors. Let X be a smooth manifold of dimension n. Let $x_0 \in X$. If $\gamma, \widetilde{\gamma} :]-1, 1[\to X$ are two C^1 curves with $\gamma(0) = \widetilde{\gamma}(0) = x_0$, we say that γ, $\widetilde{\gamma}$ are equivalent if $\|\gamma(t) - \widetilde{\gamma}(t)\| = o(t)$, $t \to 0$. (Here we choose some local coordinates $x_1, \dots, x_n$ near x_0, so that $\|\gamma(t) - \widetilde{\gamma}(t)\|$ is well defined, and we notice that the choice of local coordinates and of the corresponding norm does not influence the definition.) The equivalence class of γ will be denoted by $\gamma'(0)$ or $\frac{d}{dt}\gamma(0)$, and will be called a tangent vector at x_0. The set of all tangent vectors at a point x_0 is denoted by $T_{x_0}X$ and is called the tangent space of X at x_0. It is easy to see (by working in a system of local coordinates) that $T_{x_0}X$ is a real vector space of dimension n.

If $f, \widetilde{f} : X \to \mathbf{R}$ are two C^1 functions, we say that f, $\widetilde{f}$ are equivalent if $(f(x) - f(x_0)) - (\widetilde{f}(x) - \widetilde{f}(x_0)) = o(\|x - x_0\|)$, $x \to x_0$. We let $df(x_0)$ (called the differential of f at x_0) denote the equivalence class of f. It is (by definition) a differential 1 form at x_0, also called a cotangent vector at x_0. The set $T^*_{x_0}X$ of cotangent vectors at a point x_0 is a real vector space of dimension n. It is called the cotangent space at x_0 of X.

There is a natural duality between $T^*_{x_0}X$ and $T_{x_0}X$, given by

$$\langle df(x_0), \gamma'(t)\rangle = \left(\frac{d}{dt}\right)_{t=0} f(\gamma(t)).$$

If $x_1, \dots, x_n$ are local coordinates defined in a neighborhood of x_0, then $dx_1(x_0), \dots, dx_n(x_0)$ (or $dx_1, \dots, dx_n$ for short) form a basis of $T^*_{x_0}X$. A corresponding dual basis in $T_{x_0}X$ is given by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$, where $\frac{\partial}{\partial x_j}$ is the tangent vector induced by the curve $t \mapsto x_0 + te_j$. Here we work in the local coordinates above, and e_j denotes the jth unit vector in $\mathbf{R}^n$. It is easy to check that $df = \sum_1^n \frac{\partial f}{\partial x_j} dx_j$, $\gamma'(0) = \sum_1^n \frac{d\gamma_j}{dx_j} \frac{\partial}{\partial x_j}$ at the point x_0.

The sets $TX = \cup_{x_0\in X} T_{x_0}X$ and $T^*X = \cup_{x_0\in X} T^*_{x_0}X$ are vector bundles and in particular C^∞ manifolds. If $x_1, \dots, x_n$ are local coordinates on X, then we get the corresponding local coordinates $(x, t) = (x_1, \dots, x_n, t_1, \dots, t_n)$ on TX and $(x, \xi) = (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ on T^*X by representing $\nu \in TX$ and $\rho \in T^*X$ by their base point x (given by the coordinates $(x_1, \dots, x_n)$) and the corresponding tangent vector $\sum t_j \frac{\partial}{\partial x_j}$ and cotangent vector $\sum \xi_j dx_j$. If

$y_1, \dots, y_n$ is a second system of local coordinates, then in the intersection of the two open sets in X parameterized by the two systems of local coordinates, we have the point-wise relations $t = \frac{\partial x}{\partial y} s$, $\eta =^t (\frac{\partial x}{\partial y})\xi$ for the corresponding local coordinates (x, t), (y, s) on TX and (x, ξ), (y, η) on T^*X. Here $\frac{\partial x}{\partial y} = (\frac{\partial x_j}{\partial y_k})_{1 \le j,k \le n}$ is the standard Jacobian matrix.

If $\rho \in T^*X$, we let $\pi(\rho) \in X$ be the associated base point so that $\pi : T^*X \to X$ is the natural projection map. A section in T^*X is a right inverse of π. The same definitions can be given for TX and more generally for any vector bundle. Sections in T^*X are called differential 1 forms, and sections in TX are called vector fields. Most of the time we will only consider sections of class C^∞ and we will also most of the time only consider locally defined sections (i.e. sections of T^*U and TU where U is some small open subset of X).

A vector field can be written in local coordinates as $\nu = \sum t_j(x) \frac{\partial}{\partial x_j}$ and a differential 1 form as $\omega = \sum \xi_j(x) dx_j$.

If Y is a second manifold and $f : Y \to X$ a map of class C^1, $y_0 \in Y$, $x_0 = f(y_0) \in X$, then we have a natural map $f_* = df : T_{y_0}Y \to T_{x_0}X$, which in local coordinates is given by the ordinary Jacobian matrix $(\frac{\partial x_j}{\partial y_k})$. The adjoint is $f^* : T^*_{x_0}X \to T^*_{y_0}Y$ and we note that if u is a C^1 function on X and $\gamma : I \to Y$ is a C^1 curve, with $\gamma(0) = y_0$, 0 in the interior $\operatorname{Int} I$ of the interval I, then $(f \circ \gamma)'(0) = f_*(\gamma'(0))$, $d(u \circ f)(y_0) = f^*(du(x_0))$. More generally, if Z is a third manifold, $g : Z \to Y$ is of class C^1 and $z_0 \in Z$, $g(z_0) = y_0$, then $(f \circ g)_* = f_* \circ g_*$, $(f \circ g)^* = g^* \circ f^*$. When passing to sections, we see that if ω is a 1 form on X, then $f^*\omega$ is a well defined 1-form on Y (called the pull-back of ω by means of f). The corresponding push-forward $f_*\nu$ of a vector field ν can be defined when f is a diffeomorphism, but not in general. If $\gamma :]a, b[\to X$ is a C^1 curve and $t_0 \in]a, b[$, we recover the tangent vector $\gamma'(t_0)$ of γ at t_0 as $\gamma_*(\frac{\partial}{\partial t}(t_0))$.

The elementary theory of ordinary differential equations gives the following fact: if ν is a C^∞ vector field on X, then for every $x_0 \in X$, we can find $T_+(x_0)$, $T_-(x_0)$ in $]0, +\infty]$, such that we have a smooth (i.e. C^∞) curve:

$$\begin{aligned} &] - T_-(x_0), T_+(x_0)[\ni t \mapsto \gamma(t) =: \exp(t\nu)(x_0) \in X \\ &\text{with } \gamma(0) = x_0, \ \gamma'(t) = \nu(\gamma(t)). \end{aligned}$$

If we choose $T_\pm$ maximal, we get lower semi-continuous functions $X \ni x \mapsto T_\pm(x)$ and a smooth map

$$\{(t, x) \in \mathbf{R} \times X; -T_-(x) < t < T_+(x)\} \ni (t, x) \mapsto \Phi(t, x) = \exp(t\nu)(x)$$

with $\Phi(0, x) = x$, $\frac{\partial}{\partial t}\Phi(t, x) = \nu(\Phi(t, x))$. When t is fixed, we can in general not define $\exp t\nu$ on all of X but only on the set of $x \in X$ for which

$-T_-(x) < t < T_+(x)$. We have

$$\exp t\nu(\exp s\nu(x)) = \exp(t+s)\nu(x),$$

for t, s, x such that the left member is defined.

The canonical 1 and 2 forms. Let $\pi : T^*X \to X$ be the natural projection (which is simply $(x,\xi) \mapsto x$ in canonical coordinates). For $\rho \in T^*X$, consider $\pi^* : T^*_{\pi(\rho)}X \to T^*_\rho(T^*X)$. Since $\rho \in T^*_{\pi(\rho)}X$, we can define the canonical 1 form $\omega_\rho \in T^*_\rho(T^*X)$, by $\omega_\rho = \pi^*(\rho)$. Varying ρ, we get a smooth 1 form ω on on T^*X, which in canonical coordinates has the expression: $\omega = \sum \xi_j dx_j$.

We next recall some facts about forms of higher degree. If L is a finite dimensional real vector space, and L^* the dual space, then we have a natural duality between the k fold exterior product spaces $\wedge^k L$ and $\wedge^k L^*$, given by

$$\langle u_1 \wedge \ldots \wedge u_k, v_1 \wedge \ldots \wedge v_k \rangle = \det(\langle u_j, v_k\rangle),\ u_j \in L,\ v_k \in L^*.$$

Without repeating the definition of exterior products and exterior product spaces, it may be useful to recall that exterior products $u_1 \wedge \ldots \wedge u_k$ are linear in each of their factors and change sign if we permute two neighboring factors. Moreover, if $e_1, \ldots, e_n$ form a basis for L, then a basis for $\wedge^k L$ is formed by the $e_{j_1} \wedge \ldots \wedge e_{j_k}$, for all $1 \le j_1 < j_2 < \ldots < j_k \le n$.

If M is a C^∞ manifold of dimension m, then a differential k form v is a section of the vector bundle $\wedge^k T^*M$. In local coordinates $x_1, \ldots, x_m$:

$$v = \sum_{|I|=k} v_I(x) dx^I, \tag{1.1}$$

where in general, $I = (i_1, \ldots, i_k) \in \{1, 2, \ldots, m\}^k$, $|I| = k$, $dx^I = dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. The representation above becomes unique if we restrict the sum to the set of I with $i_1 < i_2 < \ldots < i_k$. If v is a k form of class C^1 locally given by (1.1), we define the $k+1$ form

$$dv = \sum_{|I|=k} dv_I \wedge dx^I. \tag{1.2}$$

dv is called the exterior differential of v and it can be shown that its definition does not depend on the choice of local coordinates or on the choice of the representation (1.1). We have the following facts:

$$d^2 = 0, \tag{1.3}$$

If ω is a $k+1$ form of class C^∞, which is closed in the sense that $d\omega = 0$, then in every open set in M which is diffeomorphic to a ball, we can find a smooth k form v, such that $dv = \omega$. (1.4)

If $f : Y \to X$ is a smooth map between two smooth manifolds, then there is a unique way of extending the pull-back f^* from 1 forms to k forms by multilinearity. If v is a smooth k form on X, then $d(f^*v) = f^*(dv)$. (1.5)

We now return to the canonical 1 form ω on T^*X, and define the canonical 2 form σ on T^*X as $\sigma = d\omega$. In canonical coordinates:

$$\sigma = \sum_1^n d\xi_j \wedge dx_j. \tag{1.6}$$

For $\rho \in T^*X$, $\sigma_\rho \in \wedge^2 T^*_\rho(T^*X)$ can be viewed as a linear form on $\wedge^2 T_\rho(T^*X)$ or equivalently as an antisymmetric bilinear form on $T_\rho(T^*X) \times T_\rho(T^*X)$. Mixing the two points of view, we write:

$$\sigma_\rho(t,s) = \langle \sigma_\rho, t \wedge s \rangle,\ \ t, s \in T_\rho(T^*X).$$

In canonical coordinates, and with the notation $t = (t_x, t_\xi)$ (so that $t = \sum (t_{x_j}\frac{\partial}{\partial x_j} + t_{\xi_j}\frac{\partial}{\partial \xi_j})$), $s = (s_x, s_\xi)$, we get

$$\sigma_\rho(t,s) = \langle t_\xi, s_x\rangle - \langle s_\xi, t_x \rangle = \sum (t_{\xi_j}s_{x_j} - s_{\xi_j}t_{x_j}).$$

From this it is clear that σ_ρ is a non-degenerate bilinear form and consequently there is a bijection $H : T^*_\rho(T^*X) \to T_\rho(T^*X)$ determined by:

$$\sigma(s, Hu) = \langle s, u\rangle\ s \in T_\rho(T^*X),\ u \in T^*_\rho(T^*X).$$

In canonical coordinates, if $u = u_x dx + u_\xi d\xi = \sum (u_{x_j}dx_j + u_{\xi_j}d\xi_j)$, we get $Hu = u_\xi \frac{\partial}{\partial x} - u_x \frac{\partial}{\partial \xi}$. If $f(x,\xi)$ is of class C^1 on X (or on some open subset of X), we define the Hamilton field of f by $H_f = H(df)$. In canonical coordinates,

$$H_f = \sum_1^n (\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j}).$$

If M is a manifold, $\rho \in M$, $t \in T_\rho M$, then we define the contraction $t\rfloor : \wedge^k T^*_\rho M \to \wedge^{k-1}T^*_\rho M$ as the adjoint of the left exterior multiplication $t\wedge : \wedge^{k-1}T_\rho M \to \wedge^k T_\rho M$. Then with $M = T^*X$, the Hamilton field is (equivalently) defined by the pointwise relation,

$$H_f \rfloor \sigma = -df. \tag{1.7}$$

If f, g are two C^1 functions defined on some open set in T^*X, we define their Poisson bracket as the continuous function

$$\{f, g\} = H_f(g) = \langle H_f, dg\rangle = \sigma(H_f, H_g),$$

where, in the second expression, we view H_f as a first order differential operator. In canonical coordinates,

$$\{f, g\} = \sum(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j})$$

Notice that $\{f, g\} = -\{g, f\}$, and in particular that $\{f, f\} = 0$.

Lie derivatives. Let v be a smooth vector field on a manifold M and let ω be a smooth k form on M. Then the Lie derivative of ω along v is defined pointwise by

$$\mathcal{L}_v\omega = \left(\frac{d}{dt}\right)_{t=0}((\exp tv)^*\omega).$$

If u is a another smooth vector field on M, we also define

$$\mathcal{L}_v u = \left(\frac{d}{dt}\right)_{t=0}((\exp -tv)_* u).$$

Here we need of course to observe that the push-forward of a vector field by means of a local diffeomorphism can be defined locally. We have the following facts:

(1) When ω is a 0 form, i.e. a function, then $\mathcal{L}_v\omega = v(\omega)$.

(2) $\mathcal{L}_v u = [v, u] = vu - uv$, where u, v are viewed as first order differential operators in the last two expressions.

(3) $\mathcal{L}_v(d\omega) = d(\mathcal{L}_v\omega)$,

(4) $\mathcal{L}_v(\omega_1 \wedge \omega_2) = (\mathcal{L}_v\omega_1) \wedge \omega_2 + \omega_1 \wedge (\mathcal{L}_v\omega_2)$,

(5) $\mathcal{L}_v(u\rfloor\omega) = (\mathcal{L}_v u)\rfloor\omega + u\rfloor(\mathcal{L}_v\omega)$,

(6) $\mathcal{L}_v\omega = v\rfloor d\omega + d(v\rfloor\omega)$,

(7) $\mathcal{L}_{v_1+v_2} = \mathcal{L}_{v_1} + \mathcal{L}_{v_2}$.

Lemma 1.1. *If f is a smooth function on some open subset in T^*X, then $\mathcal{L}_{H_f}\sigma = 0$.*

Proof. It suffices to make the calculation,

$$\mathcal{L}_{H_f}\sigma = H_f \rfloor d\sigma + d(H_f \rfloor \sigma) = H_f \rfloor d^2\omega - d^2 f = 0.$$

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Locally, we can define the maps $\Phi_t = \exp tH_f$, when $|t|$ is sufficiently small and we have $\Phi_t^*\sigma = \sigma$. In fact, we have pointwise:

$$\frac{d}{dt}\Phi_t^*\sigma = (\frac{d}{ds})_{s=0}\Phi_t^*\Phi_s^*\sigma = \Phi_t^*\mathcal{L}_{H_f}\sigma = 0.$$

Lemma 1.2. *If f, g are two smooth functions defined on some open subset of T^*X, then $[H_f, H_g] = H_{\{f,g\}}$.*

Proof. We have to show that $[H_f, H_g]\rfloor\sigma = -d\{f, g\}$. This follows from the computation:

$$\begin{aligned} -d\{f,g\} &= -d(\mathcal{L}_{H_f}g) = -(\mathcal{L}_{H_f}dg) = \mathcal{L}_{H_f}(H_g\rfloor\sigma) \\ &= [H_f, H_g]\rfloor\sigma + H_g\rfloor(\mathcal{L}_{H_f}\sigma) = [H_f, H_g]\rfloor\sigma. \end{aligned}$$

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Using the preceding lemma, it is easy to prove the Jacobi identity for three smooth functions,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Lagrangian manifolds. A submanifold $\Lambda \subset T^*X$ is called a Lagrangian manifold if $\dim\Lambda = \dim X$ and $\sigma_{|\Lambda} = 0$. In general, we define the restriction of a differential k form to a submanifold as the the pull-back of this form by means of the natural inclusion map, and there is a corresponding natural way of viewing the tangent space of a submanifold at some given point as a subspace of the tangent space of the ambient manifold at the same point. If Λ is a submanifold of T^*X and $\rho \in \Lambda$, then we define $T_\rho\Lambda^\perp \subset T_\rho(T^*X)$ as the orthogonal space with respect to σ of $T_\rho\Lambda \subset T_\rho(T^*X)$. The sum of the dimensions of $T_\rho\Lambda$ and $T_\rho\Lambda^\perp$ add up to the dimension of $T_\rho(T^*X)$, but there is no reason for $T_\rho\Lambda$ and $T_\rho\Lambda^\perp$ to have zero intersection. As a matter of fact, it is clear that a submanifold $\Lambda \subset T^*X$ is Lagrangian if and only if $T_\rho\Lambda = T_\rho\Lambda^\perp$ for every $\rho \in \Lambda$. That there are plenty of Lagrangian submanifolds follows from the following result.

Theorem 1.3. *Let $\Lambda \subset T^*X$ be a submanifold with $\dim \Lambda = \dim X$ and such that $\pi_{|_\Lambda} : \Lambda \to X$ is a local diffeomorphism (in the sense that every point ρ in Λ has a neighborhood in Λ which is mapped diffeomorphically by π onto a neighborhood of $\pi(\rho)$). Then Λ is Lagrangian if and only if for each point $\rho \in \Lambda$, we can find a (real) C^∞ function $\phi(x)$ defined near $\pi(\rho)$, such that Λ coincides near ρ with the manifold $\{(x, d\phi(x)); x \in$ some neighborhood of $\pi(\rho)\}$.*

Proof. If ω is the canonical 1 form, we notice that $d(\omega_{|_\Lambda}) = \sigma_{|_\Lambda}$. Therefore the following three statements are equivalent:

(1) Λ is Lagrangian.

(2) $\omega_{|_\Lambda}$ is closed (i.e. $d(\omega_{|_\Lambda}) = 0$).

(3) Locally on Λ, we can find a smooth function ϕ with $\omega_{|_\Lambda} = d\phi$.

If $x_1, \ldots, x_n$ are local coordinates on X, we can also view them (or rather their compositions with π) as local coordinates on Λ, and represent Λ by equations $\xi = \xi(x)$ in the corresponding canonical coordinates. Then (3) is equivalent to $\xi_j(x) = \frac{\partial \phi(x)}{\partial x_j}$, i.e. $\sum \xi_j(x) dx_j = d\phi$. #

Hamilton–Jacobi equations. These equations are of the form $p(x, \phi'_x) = 0$, where p is a real-valued C^∞ function defined on some open subset of T^*X. Here we shall also assume that $dp(x,\xi) \neq 0$, when $p(x,\xi) = 0$. The basic idea in treating a Hamilton-Jacobi equation is to consider the Lagrangian manifold $\Lambda = \Lambda_\phi$ associated with ϕ as in the preceding theorem, and try to find such a manifold inside the hypersurface H defined by $p(x,\xi) = 0$. If $\rho \in \Lambda$, we shall then have $T_\rho\Lambda \subset T_\rho H$ (considering these tangent spaces as subspaces of $T_\rho T^*X$), and hence $T_\rho^\perp H \subset T_\rho\Lambda$, since $T_\rho\Lambda^\perp = T_\rho\Lambda$. It is easy to see that $T_\rho H^\perp = \mathbf{R}H_p$, so we must have $H_p \in T_\rho\Lambda$ at every point $\rho \in \Lambda$, or in other words H_p should be tangent to Λ at every point of Λ.

Proposition 1.4. *Let $\Lambda' \subset H$ be an isotropic submanifold (in the sense that $\sigma_{|_{\Lambda'}} = 0$) of dimension $n-1$ passing through some given point $\rho_0 \in H$ and such that $H_p(\rho_0) \notin T_{\rho_0}\Lambda'$. Then in a neighborhood of ρ_0 we can find a Lagrangian manifold Λ such that $\Lambda' \subset \Lambda \subset H$ (in that neighborhood).*

Proof. According to the observation above it is natural to try

$$\Lambda = \{\exp(tH_p)(\rho);\ |t| < \epsilon,\ \rho \in \Lambda',\ |\rho - \rho_0| < \epsilon\}$$

for some sufficiently small $\epsilon > 0$. (Here $|\rho - \rho_0|$ is well-defined, if we choose some local coordinates.) Then $\Lambda' \subset \Lambda$ (near ρ_0) and since H_p is tangent to

H (by the relation $H_p p = 0$) we also have $\Lambda \subset H$. From the assumption $H_p(\rho_0) \notin T_{\rho_0}\Lambda'$ and the implicit function theorem, it also follows that Λ is a smooth manifold of dimension n.

In order to verify that Λ is Lagrangian, we first take $\rho \in \Lambda'$ (with $|\rho - \rho_0| < \epsilon$) and consider $T_\rho\Lambda = T_\rho\Lambda' \oplus \mathbf{R}H_p$. Then $\sigma_{\rho|_{T_\rho\Lambda\times T_\rho\Lambda}} = 0$ since $\sigma_{\rho|_{T_\rho\Lambda'\times T_\rho\Lambda'}} = 0$, $\sigma_\rho(H_p, H_p) = 0$, $\sigma_\rho(t, H_p) = \langle t, dp\rangle = 0$ for all $t \in T_\rho\Lambda' \subset T_\rho H$.

More generally, at the point $\rho_t = \exp(tH_p)(\rho)$, $\rho \in \Lambda'$, we have

$$T_{\rho_t}(\Lambda) = \exp(tH_p)_*(T_\rho\Lambda)$$

and for $u, v \in T_\rho\Lambda$ we get, using the fact that $\exp(tH_p)^*\sigma_{\rho_t} = \sigma_\rho$:

$$\sigma_{\rho_t}(\exp(tH_p)_*u, \exp(tH_p)_*v) = \sigma_\rho(u, v) = 0.$$

We have then verified that $\sigma_{|_\Lambda} = 0$, which suffices since Λ has the right dimension. #

In the following we write $x = (x', x_n) \in \mathbf{R}^n$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$.

Theorem 1.5. *Let $p(x,\xi)$ be a real valued C^∞ function, defined in a neighborhood of some point $(0,\xi_0) \in T^*\mathbf{R}^n$, such that $p(0,\xi_0) = 0$, $\frac{\partial p}{\partial \xi_n}(0,\xi_0) \neq 0$. Let $\psi(x')$ be a real valued C^∞ function defined near 0 in $\mathbf{R}^{n-1}$ such that $\frac{\partial \psi}{\partial x'}(0) = \xi_0'$. Then there exists a real valued smooth function $\phi(x)$, defined in a neighborhood of $0 \in \mathbf{R}^n$, such that in that neighborhood:*

$$p(x, \phi_x'(x)) = 0,\ \ \phi(x', 0) = \psi(x'),\ \ \phi_x'(0) = \xi_0. \tag{1.8}$$

If $\widetilde{\phi}(x)$ is a second function with the same properties, then $\phi(x) = \widetilde{\phi}(x)$ in some neighborhood of 0.

Proof. In a suitable neighborhood of $(0,\xi_0) \in \mathbf{R}^{n-1} \times \mathbf{R}^n$ we have $p(x', 0, \xi) = 0$ if and only if $\xi_n = \lambda(x', \xi')$, where λ is a real valued C^∞ function, with $\lambda(0, \xi_0') = (\xi_0)_n$. Let

$$\Lambda' = \{(x,\xi);\ x_n = 0,\ \xi' = \frac{\partial \psi}{\partial x'}(x'),\ \xi_n = \lambda(x', \xi'),\ x' \in \mathrm{neigh}\,(0)\},$$

where neigh (0) indicates some sufficiently small neighborhood of 0.

Then $\Lambda' \subset p^{-1}(0)$ is isotropic of dimension $n-1$ and H_p is nowhere tangent to Λ' since H_p has a component $\frac{\partial p}{\partial \xi_n}\frac{\partial}{\partial x_n}$ with $\frac{\partial p}{\partial \xi_n} \neq 0$. Let $\Lambda \subset p^{-1}(0)$ be a Lagrangian manifold as in Proposition 1.4. The differential of $\pi_{|_\Lambda} : \Lambda \to \mathbf{R}^n$ is bijective at $(0,\xi_0)$, so if we restrict the attention to a sufficiently small neighborhood of that point, we can apply Theorem 1.3 and see that Λ is of the form $\xi = \phi'(x)$, $x \in \mathrm{neigh}\,(0)$. We have then $p(x, \phi'(x)) = 0$, $\phi'(0) = \xi_0$. Since $\Lambda' \subset \Lambda$, we get $\frac{\partial \psi}{\partial x'}(x') = \frac{\partial \phi}{\partial x'}(x', 0)$, so modifying ϕ by a constant, we get $\phi(x', 0) = \psi(x')$. We leave the verification of the uniqueness statement as an exercise. #

We can view Λ as a union of integral curves of H_p, passing through Λ'. The projection of such an integral curve is an integral curve of the field $\nu = \sum_1^n \frac{\partial p}{\partial \xi_j}(x, \phi'_x(x)) \frac{\partial}{\partial x_j}$, which can be identified with $H_{p_{|\Lambda}}$ via the projection $\pi_{|\Lambda}$. If $q(x,\xi) = \sum_1^n \frac{\partial p}{\partial \xi_j}(x,\xi)\xi_j$, we have the trivial identity

$$\sum_1^n \frac{\partial p}{\partial \xi_j}(x, \phi'_x(x)) \frac{\partial}{\partial x_j}\phi = q(x, \phi'_x(x)).$$

If $x = x(t)$ is an integral curve of ν with $x_n(0) = 0$, then we get $\phi(x(t)) = \psi(x'(0)) + \int_0^t q(x(s), \xi(s))ds$, where $\xi(s) = \phi'(x(s))$, so that $s \mapsto (x(s), \xi(s))$ is the integral curve of H_p with $x_n(0) = 0$, $\xi'(0) = \frac{\partial \psi}{\partial x'}(x'(0))$, $\xi_n(0) = \lambda(x'(0), \xi'(0))$.

If $\psi = \psi_\alpha$ depends smoothly on some parameters $\alpha \in \mathbf{R}^k$, then $\phi = \phi(x, \alpha)$ will be a smooth function of (x, α), and differentiating the equation $p(x, \phi'_x(x)) = 0$, we see that $\frac{\partial \phi}{\partial \alpha}$ is constant along the bicharacteristics curves, i.e. the x-space projections of the H_p integral curves in Λ_ϕ.

In order to recall the roots of symplectic geometry in classical mechanics, let us consider the case when $p = \frac{1}{2m}\xi^2 + V(x)$, where $V(x)$ is some smooth real potential and $m > 0$ is a constant. The equations for the H_p integral curves are $x'(t) = \frac{\xi(t)}{m} =: v$, $\xi'(t) = -V'(x(t))$, and if we eliminate $\xi(t)$, we get the differential equation for $x(t)$: $x''(t) = -\frac{V'(t)}{m}$. We can view this as the motion of a classical partical of mass m. ξ is the momentum, so that $\xi(t) = mv(t)$. $v(t)$ is the velocity. The total energy $\frac{1}{2m}\xi^2 + V(x)$ is a constant of the motion (i.e. constant on every integral curve). Finally $-\frac{V'(x)}{m}$ is the acceleration induced by the force $-V'(x)$.

Another motivation for working on the cotangent bundle comes from the general theory of partial differential equations. Consider a differential operator with smooth coefficients, $P : C^\infty(X) \to C^\infty(X)$, where X is a manifold. For every choice of smooth local coordinates, P takes the form $P = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha$, where $m \in \mathbf{N} = \{0, 1, 2, \dots\}$ is the order of the operator and we use standard multi-index notation: $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D_x = (D_{x_1}, \dots, D_{x_n})$, $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$. In the corresponding canonical coordinates (x, ξ), we define the principal symbol,

$$p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \text{ where } \xi^\alpha = \xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}.$$

Then one can check that $p(x,\xi)$ is a well-defined function in $C^\infty(T^*X)$. A quick way to see this is to notice that if a, $\phi \in C^\infty(X)$ and ϕ is real valued, then

$$P(a(x)e^{i\lambda\phi(x)}) = e^{i\lambda\phi(x)}(p(x,\phi'(x))\lambda^m a(x) + \mathcal{O}(\lambda^{m-1})), \text{ when } \lambda \to \infty.$$

Notes

The presentation here is close to that of [GrSj] and has been inspired by that of Duistermaat [Du]. The reader who wants to study the subject in depth could turn to the books of Sternberg [St] and Hofer–Zehnder [HoZe].