POLYHEDRA

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"One of the most charming chapters of geometry"

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Preface

Polyhedra have been a part of the fabric of mathematics for two thousand years and have been the inspiration for contributions to many branches of the subject. So it seems remarkable that information concerning their history and mathematical properties is quite difficult to find. The study of polyhedra is still an active area of research and, along with other parts of geometry, is currently enjoying something of a renaissance. However, it is still possible (even probable) that students can complete their education to graduate level and not meet such fundamental objects as the five Platonic solids. The lack of adequate sources of information may contribute to this state of affairs. This book is my response to this vacuum. It tells the story of what people have thought about polyhedra over the ages, and explains some of the mathematics that has been developed to study them.

I started the project that developed into this book by chance over seven years ago. I had just completed my Ph.D. studies at Liverpool University, where the mathematics department had recently begun to build up a collection of mathematical exhibits. These included models of all the polyhedra labelled regular: the five Platonic solids, the four star polyhedra, and five compounds. Each week, one departmental seminar was set aside for general interest talks given by members of staff and research students so that people could find out what their colleagues were thinking about. In one of these sessions I decided to explain some of the mathematical properties of the new polyhedral models. Naively expecting to find all the information I needed presented in several easily available books, I visited the library. The books I found were of three types. Some contained what is often referred to as 'recreational mathematics'. These books often had a chapter or two on the basic properties of a few kinds of polyhedra. The more advanced of such books mentioned Euler's formula. The second category of books fell at the other end of the spectrum. They concentrated on polytopes in arbitrary dimensions and occasionally dealt with three-dimensional examples. The other books I found were guides for model-making. Even large volumes on the history of mathematics contained few comments on the subject of polyhedra after the Greeks, presumably because it is not seen as part of 'mainstream mathematics'.

I gleaned all the information I could from these sources, followed up their references to journal articles and gradually built up enough material to present a survey of the area. I supplemented the display models with some from my own collection and talked on 'The history of polyhedra'. After the seminar several colleagues asked for my sources of information. On hearing of the trouble I had had they suggested that I write up the talk. Since then I have searched for information to fill in some of the details of my original talk and this book is the result.

I am not a historian and I doubt whether the book should be regarded as a history of polyhedra. I have simply collected together the things that I found interesting and arranged them by theme in approximate chronological order. My aim was to present more than a catalogue of definitions and theorems. The dry predigested style commonly used in textbooks tends to ignore the motivation that led to the results being described and leaves readers confused by details they see no need for. I have tried to place results in context, to trace the development of the underlying ideas, and find their influences on, and connections with, other subjects both within mathematics and further afield.

My selection of topics is, of course, a personal choice. Although the work is certainly not encyclopaedic, I think the coverage is fairly complete up to the turn of the century. I have chosen to concentrate on the three-dimensional, geometric aspects of the subject since a large part of the attraction of polyhedra derives from making and experimenting with models. Very few topics that could be recast as graph theory have been included. This means that many developments made this century on the combinatorial properties of polyhedra are not covered. There are three major omissions of this kind that I am aware of. The first is Steinitz' theorem, which plays a central role in connecting the geometric and combinatorial sides of the subject. A presentation of this theorem is easily available in Branko Grünbaum's book *Convex Polytopes*. A discussion of Eberhard-type theorems and Alexandrov's theorem is also missing. Perhaps more surprising to some readers will be the omission of references to duality. Although we can retrospectively find the concept in much early work, it is not clear whether the authors were aware of it. In any case, what do we mean by duality? A few passing remarks on the reciprocal properties of the Platonic solids could cause more confusion than insight: projective and combinatorial duality have been muddled for ages. Furthermore, we do not always find duality in places where it might be expected. For instance, there is no way to construct the vertex-transitive polyhedra from the face-transitive ones 'by duality'. To discuss the topic fully would take a chapter in itself.

Other topics that are not mentioned are space-filling or neighbourly polyhedra, connections with higher-dimensional polytopes or tilings, woven polyhedra of various kinds, polyhedral embeddings of mathematically interesting surfaces, or applications to linear programming and computational geometry.

Skimming through the book you will see it contains many illustrations. I think these are essential when discussing a subject so many of whose underlying intuitions derive from visual experience. The book also contains proofs. I have tried to explain these simply but fully. For the most part they are self-contained. One exception to this rule is the application of some group theory in the last two chapters. The theorem which is appealed to is discussed in Appendix 2.

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Peter Cromwell Liverpool, 1996