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978-0-521-66405-9 - Polyhedra: "One of the Most Charming Chapters of Geometry"
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Excerpt
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Introduction

Geometry is a skill of the eyes and the hands as well as of the mind.

J. Pederson

Models of polyhedra adorn the personal spaces of people with a broad range of mathematical experience. In the university office of the professional mathematician, the teacher's classroom, and the child's bedroom, these attractive geometrical objects have a universal appeal. Their popularity has endured for centuries. This book explores how the study of polyhedra has developed, the ways people have used them and thought about them over the ages, and how their ideas have evolved.

No science advances in a smooth and continuous way and the study of polyhedra is no exception. We follow some of the searches for explanations of observed or conjectured properties of polyhedra. Sometimes this is a frustrating struggle to understand the foundations and limitations of a new concept. What we might retrospectively regard as a significant milestone is often the result of the accumulated effort of many people over a long period. Progress can also be made in rapid leaps as a fresh mind brings a brilliant insight to an old problem which then throws open a whole new region for exploration.

We shall see that the study of polyhedra has contributed to several areas of mathematics and has connections with many others. However, polyhedra are not confined to mathematics. Anything which is bounded by flat surfaces and which has well-defined corners has a polyhedral form. It is easy to think of examples from architecture. Many more can be found in the three kingdoms of nature—mineral, vegetable and animal. Polyhedra have been used in philosophical or scientific explanations of the world around us. They have even found their way into art, literature and theological debate. We will begin with a brief look at some of these.

Polyhedra in architecture

Examples of polyhedra in architecture are easy to find. The ancient pyramids at Giza in Egypt, built over four and a half thousand years ago, are probably the most simple in design. Modern office blocks are often prismatic structures of steel and glass. Other buildings combine both elements, placing a pyramidal roof structure on a prismatic living space. An octagonal version of this principle underlies the Baptistry at Florence in Italy.

More unusual polyhedral constructions have been used for apartments in various parts of the world. A complex of cubes with their diagonals placed vertically so that each rests on a corner has been erected in Rotterdam in Holland. In Israel, apartments have been made from a complex of dodecahedra. The geodesic domes invented by R. Buckminster Fuller in the 1940’s are some of the most remarkable forms of polyhedral architecture. These first impressed the world at the *Expo 67* world fair held at Montreal in Canada. They are now used to protect astronomical telescopes and radio antennae from the elements. On a smaller scale they are used as the frameworks for hemispherical glasshouses and tents, and children’s climbing frames.

Polyhedra in art

Polyhedra became popular motifs in art when linear perspective was introduced by the Italians of the fifteenth century. The flat faces and hard edges of polyhedral forms make them very good exercises for those wanting to practice perspective constructions, and many painters’ manuals written in the Renaissance include instructions for foreshortening the Platonic solids. Finished paintings sometimes included polyhedra as ornaments but they were usually disguised as pavilions, architecture or headwear.

Polyhedra also appear in twentieth-century art. The *Sacrament of the Last Supper* by Salvador Dali contains a skeletal outline of part of a regular dodecahedron—the Platonic symbol of the universe. Several works by the Dutch graphic artist M. C. Escher contain polyhedra, often star polyhedra or compounds. Op-art designs include many apparently flat-faced objects which cannot be given a consistent three-dimensional interpretation. These can be regarded as ‘impossible polyhedra’. A well-known object of this kind is the Penrose tribar shown in Figure I.1.

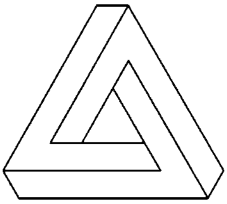

Figure I.1. The Penrose tribar—an ‘impossible’ polyhedron.

The same properties of polyhedra which made them attractive to the first artists who used perspective techniques (the fact the positions of a few vertices give enough information to completely describe the solid) make them appealing as computer graphics. Computers can manipulate and draw pictures of simple polyhedra very quickly. More complex polyhedral meshes are used by computer-aided design packages to assist engineers create the bodies of new cars and aircraft.

In the science-fiction film *Tron* a man finds himself transported into a computer where he meets various programs and other elements of computer architecture. In one of the more friendly encounters, he has a conversation with a floating polyhedron that continually changes shape. After having all his questions answered either 'yes' (when the polyhedron metamorphoses into a large yellow regular octahedron) or 'no' (as it becomes an orange stellation of the icosahedron) he realises that this object is the most basic element inside the machine—a bit.

Much modern abstract sculpture has a polyhedral form. Sometimes this is as simple as a cube with one corner embedded in the ground. Other sculptors use tetrahedra stuck together face to face to make vertical columns or sprawling snakelike creatures. One of the largest polyhedral sculptures is in Vegreville, Alberta, Canada. It is a huge Easter egg over seven metres high designed and built by Ronald Dale Resch. It is constructed from 2732 bronze, silver and gold tiles some eighty percent of which are equilateral triangles. The others are non-convex equilateral hexagons in the shape of three-pointed stars. The variations in the egg's curvature are achieved by changing the angles in the stars.

Polyhedra in ornament

Many ornaments have polyhedral forms. Vases and decorative containers are an obvious source. A commonly used shape is the cub-octahedron, probably because it is so easy to make. Earrings of this shape dating from 450AD have been found in Germany. Cub-octahedra are also found in the decorations on Japanese shrines. A large cub-octahedron embellished with chrysanthemums, the emblem of the emperor, sits on top of a tea house in the Shugakuin Imperial Villa in Kyoto. Sacred lanterns of this shape have been made since the thirteenth century and are still used in ceremonies to commemorate the dead. The Koreans use rhomb-cub-octahedral lanterns.

Around fifty bronze ornaments or charms of dodecahedral shape have survived from Roman times and can be seen in the museums around Europe. Most are hollow with circular holes of various sizes cut in their faces and small balls attached at their vertices. Some older examples of Etruscan origin are also known. A dodecahedron recently discovered in Switzerland has a lead core covered with silver and the names of the signs of the zodiac inscribed on its faces. The association of the twelve months with the dodecahedron still continues. One issue

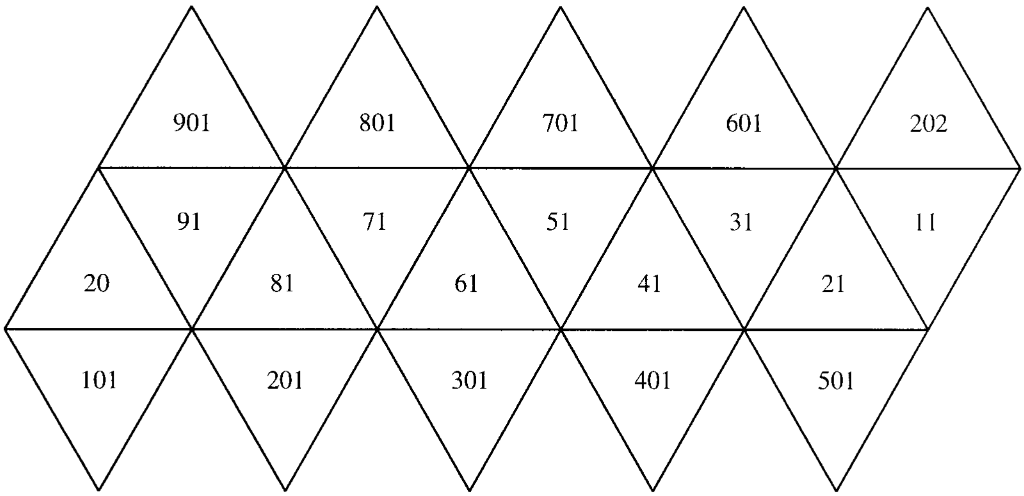

Figure I.2. Net for the Tipu Sultan icosahedron.

each year of *Mathematics in School*, a magazine for teachers, prints the net of a dodecahedron with a calendar printed on it—one month per face.

Dice are another common form of polyhedral object. All the regular solids have been used as dice. A curious icosahedral die was found in the treasure of Tipu Sultan in India when he was overthrown by the British in 1799. It is made of gold and has an unusual distribution of numbers on it. A net is shown in Figure I.2. To obtain a die with ten faces both dipyramids and prisms have been used. A very unusual die in the shape of a rhomb-cub-octahedron was unearthed at Corfe Castle, southern England, in 1973. It is made from a local black marble and is thought to be between two and three hundred years old. Only the square faces have markings on them. Pairs of letters are incised in six of them and the other squares contain patterns of circles representing the first twelve integers. Figure I.3 contains a net for this solid. Its use is unknown.

Polyhedra in nature

The precious gemstones cut and set into rings are sparkling examples of polyhedra but their facets are produced artificially. Natural processes can produce equally striking results: crystals (Plate 2). Bounded by flat reflective planes, their obvious geometric features contrast strongly with the rounded, soft, flexible or irregular qualities more frequently found in natural forms. Because of this distinctiveness, they have always attracted attention. In the nineteenth century, the study of polyhedra and crystals led to the geometric analysis of symmetry. Symmetry theory, together with the assumption that crystals are built up as repeating arrays of atoms, implies the crystallographic restriction: crystals can only have two-fold,

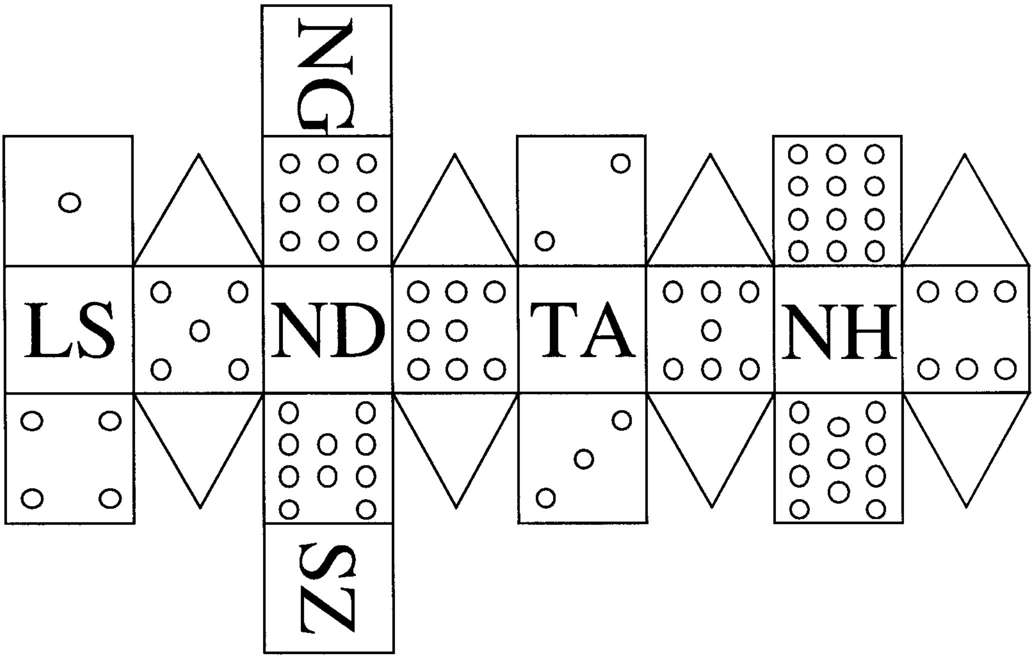

Figure I.3. Net for the Corfe Castle rhomb-cub-octahedral die.

three-fold, four-fold or six-fold rotational symmetry. For this reason, the discovery in 1984 of a crystalline-looking substance with five-fold symmetry caused great excitement. These objects are now called *quasi*crystals.

The kernels of some nuts and fruits contain many small seeds which grow in a restricted space. Pomegranates are one example. As each seed grows it presses up against its neighbours. The seeds prevent each other from expanding uniformly and they grow to fill the available space producing flat-faced seeds with sharp corners. If the seeds had a perfectly uniform distribution before they began to grow and were subjected to isotropic compression forces they would end up as rhombic dodecahedra.

The principal of economy—maximising volume from given materials—leads to the construction of roughly spherical organisms. These sometimes have polyhedral substructures. Ernst Haeckel on his voyage on H.M.S. Challenger, in the 1880's, drew many pictures of microscopic single-celled creatures called radiolaria. A radiolarian has a spherical skeleton that is polyhedral in character. Haeckel named three of them *circoporus octahedrus*, *circorrhegma dodecahedra* and *circogonia icosahedra* because he thought they resembled the Platonic solids. His illustrations are shown in Figure I.4.

Spherical cages also form part of simple viruses such as that responsible for polio. Viruses reproduce themselves by taking over the protein synthesising equip-

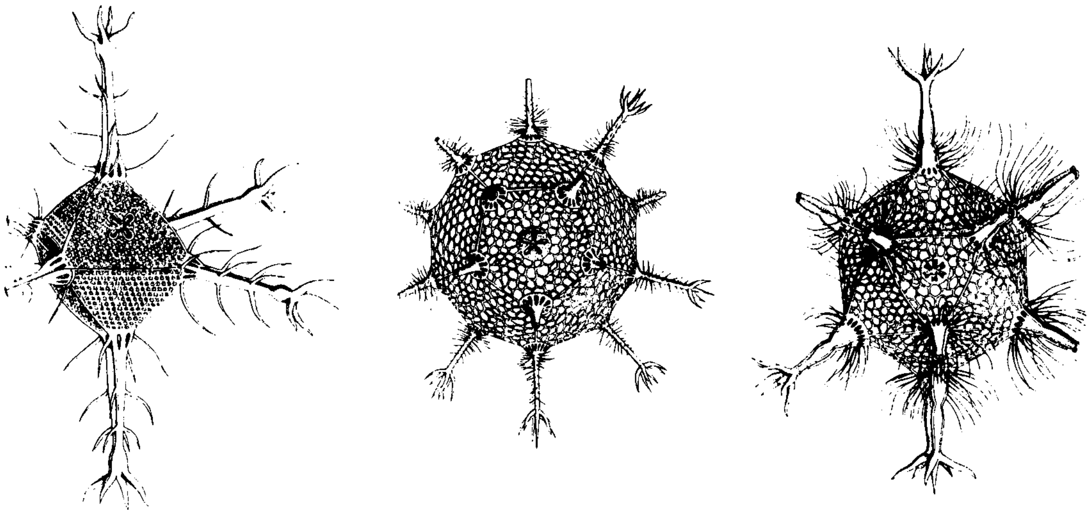

Figure I.4. Three of Haeckel's sketches of radiolaria.

ment of living cells. The viral nucleic acid introduced into the cell causes the cell's machinery to produce parts for new protein cages which protect the replicated RNA in the new viruses. The simplest virus cages are built up from repeating units that clump together in groups of five (pentamers) or six (hexamers). These pentagons and hexagons then fit together to form spherical capsules with approximate icosahedral symmetry.

The recently discovered allotrope of carbon also forms polyhedral spheres, ellipsoids and tubes. In the smallest example, C_{60}, the sixty atoms are arranged in the same pattern as the vertices of a truncated icosahedron—familiar as a soccer ball. These carbon cages have been named Fullerenes in honour of Buckminster Fuller but they are colloquially known as 'Bucky balls'.

Polyhedral molecules have been known for some time. Organic chemists have made carbon–hydrogen structures such as cubane, C_8H_8, whose carbon atoms lie at the corners of a cube. Many more examples occur in inorganic chemistry, particularly with compounds involving the transition metals. In a molybdenum chloride ion ($Mo_6Cl_8^{4+}$) the chlorine atoms form a cubic cage around an octahedron composed of metal atoms. Halides of platinum and zirconium provide further examples of molecular polyhedra. Compounds of boron and hydrogen, called *boranes*, have triangular-faced polyhedral structures which include some of the deltahedra. Molecules of the borane B_8H_8 oscillate back and forth between the forms of a Siamese dodecahedron and a square antiprism.

Polyhedra in cartography

Making maps of the world has been a problem ever since we discovered that the Earth is not flat. A globe is spherical and can be used to represent the world accurately but it provides a limited view: we cannot study the whole of the Earth's surface at the same time. Transferring data from a sphere to a flat surface presents great difficulties and always results in some distortion. In the commonly seen Mercator projection, invented in the sixteenth century, a cylinder is placed around the globe so that the two surfaces touch along the equator. The features on the surface of the sphere are then projected outwards until they meet the cylinder. The map is an exact representation along the line of contact, but away from the equator the map is less precise. The distortion is worse nearest the poles; the poles themselves cannot be represented.

Buckminster Fuller was frustrated by this inaccurate view of the world in which Greenland appears three times as large as South America when, in reality, the opposite is the case. Distortion must appear in any flat map of the Earth but it might be possible to smear it out evenly so that it is less noticeable. Fuller sought to show the shape, distribution, and relative sizes of the Earth's landmasses while containing the worst distortion to the seventy-five percent of the Earth's surface covered by water.

He took a regular solid, the icosahedron, composed of twenty equilateral triangles and subdivided each face into smaller triangles. Calculating from a similar grid superimposed on the Earth he could transfer the data from the sphere to the polyhedron. This was all done between the two world wars before computers were available to assist with the calculations. The resulting map is unique in its lack of visible distortion. Fuller also had to choose the position of the icosahedron carefully so that it could be cut along its edges and opened out flat without creating unnatural breaks in any landmass. The result, known as the *Dymaxion map*, is shown in Figure I.5.

In February 1943, *Life* magazine included a colour, cut-out version of Fuller's map. Before it was published, the editors had the map examined by a panel of experts to certify that it was an accurate representation of the Earth and that it was a new discovery. The panel, which included the Chief Geographer of the U.S. State department and two mathematicians, could not find geographical or mathematical flaws in Fuller's map but were still uncertain as to how it had been created. In terms filled with negative connotations, their report concluded that it was 'pure invention'. When Fuller applied for a patent for his map he found a ruling had been issued which decreed all possible projection methods used in cartography had been exhausted. Hence his application was rejected. He presented the Patent Office with the *Life* report. The testimony of the experts could not be argued with and he was granted the first patent this century for innovation in map-making.

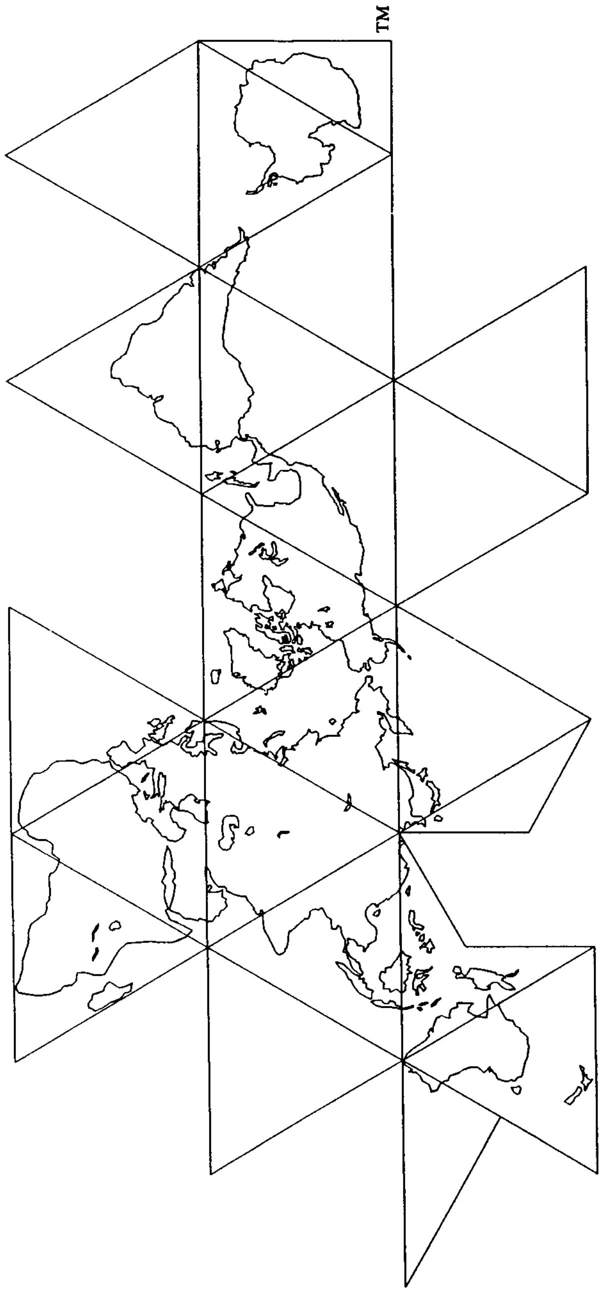

Figure I.5. Fuller's Dymaxion map

Polyhedra in philosophy and literature

Polyhedra have played roles in theories of the universe. Both Plato and Kepler made use of the five regular solids. For Kepler, the polyhedra determined the size of the known universe, the number and relative distances of the planets. Plato associated the solids with the four Empedoclean elements and the heavens, and tried to explain the properties of matter.

Following Isaac Newton's description of the 'clockwork universe' the Design Argument for the existence of God became popular: since we see evidence of design all around us, there must be a designer. The *Natural Theology* of William Paley (1743–1805) argues this case. In his argument against the existence of an inherent, universal ordering principle, Paley used the Platonic solids as the archetypal example of order.

> *Order is not universal; which it would be if it issued from a constant and necessary principle ⋯. Where order is wanted we find it; where order is not wanted, i.e. where, if it prevailed, it would be useless, there we do not find it. ⋯ No useful purpose would have arisen from moulding rocks and mountains into regular solids, or bounding the channel of the ocean by geometrical curves.* [a]

In what must be the earliest work of science-fiction, *L'Autre Monde ou Les États et Empires de la Lune et du Soleil*, Cyrano de Bergerac (1619–1655) writes about a space flight. He was imprisoned in a tower room and hatched a plan to escape. A friend brought him materials and he constructed a flying machine which he hoped would carry him to his friend's estate. The craft was light and strong, a crystal globe having many facets, a ball like a blazing mirror, in the shape of an icosahedron! It worked by catching as much light as possible to create a void which sucked in air and thereby carried the machine upwards. However, the power of the sun's rays was greater than he anticipated and instead of landing safely outside his prison, he was carried beyond the Earth's atmosphere and towards the sun. After four months he landed on a sunspot and began an adventure.

About this book

The chapters of this book are a series of related essays, each of which explores a particular theme. They are arranged in approximate chronological order but are largely independent units that can be read in any sequence. The only exception to this is Chapter 8 which develops the ideas and notation for describing symmetry. This notation is used in the following two chapters.

As in a piece of music, some themes appear several times in the book, but each entry is slightly different from the last. The subject is modified and developed or treated from a new viewpoint which brings previously hidden elements to the fore. One such topic is regularity. Regular polyhedra were introduced by the Greeks more than two thousand years ago as part of their study of solid geometry. They knew the five solids named after Plato. When Johannes Kepler experimented with new ways of constructing polyhedra, he found two more which could be labelled regular. Two centuries later, Louis Poinsot rediscovered them and found another two. The mathematical development of symmetry led to a new way of interpreting regularity which had wider applications to compound polyhedra.

Other themes which recur throughout the book are a part of every branch of mathematics—the problems of definition, classification and enumeration. What objects are we talking about? When are two objects to be regarded as the same? What kinds of object are there and how many are there? Is there any pattern or structure in such a diverse collection?

Even though the study of polyhedra is one of oldest branches of mathematics, the theory is still being advanced. However, this does not mean that it is necessary to scale high barriers of intimidating mathematics erected over the past two thousand years to attain some understanding of what has been achieved and what is being investigated today. Because of the geometrical nature of the subject, many of the results are accessible to non-specialists. Geometry has a strong visual component. It is this which gives the aesthetic appeal to the subject and which means many ideas can be communicated with the aid of pictures or models. Even though there are many illustrations in this book, sometimes they may not be sufficient. At a few places I have suggested that you make your own models to enhance your understanding of a particular point. Hands-on experience with actual three-dimensional polyhedra is the best way to find out what is going on. Once intuition is developed the pictures may suffice as reminders.

The inclusion of proofs

Although this book tells a story, it also includes proofs of the theorems. Not everyone will want to delve into the fine details of every proof, but the ideas and arguments presented in them are a part of the story. It is for the understanding provided by these explanations that people have worked so hard.

Creating a proof is like taking something apart to see what makes it work. A good proof well explained shows not only *that* something is true but also *why* it has to be true. Unfortunately, the language of rigorous argument can be intimidating. The reasons for some of the pedantry are often lost on the reader, and a mass of details sometimes obscures the flow of ideas. Remember that a proof is just an argument designed to convince its intended audience of the truth of some statement. As audiences have become more demanding over time, so the standards required have been raised. When the oversights in yesterday's proofs