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V. S. Varadarajan

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AN INTRODUCTION TO

*Harmonic analysis on
semisimple Lie groups*

V.S. VARADARAJAN

*Department of Mathematics
University of California at Los Angeles*



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CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521663625

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First published 1989
First paperback edition (with corrections) 1999

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

Varadarajan, V.S.

An introduction to harmonic analysis on semisimple Lie groups,
V.S. Varadarajan.

p. cm. – (Cambridge studies in advanced mathematics; 16)

Bibliography: p.

Includes index.

ISBN 0 521 34156 6

1. Harmonic analysis. 2. Semisimple Lie groups.

3. Representations of groups. I. Title. II. Series.

QA403. V37 1988

515'.2433–dc19 87-30911

ISBN 978-0-521-34156-1 Hardback

ISBN 978-0-521-66362-5 Paperback

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See what a lovely shell,
Small and pure as a pearl,
Lying close to my foot,
Frail, but a work divine,
Made so fairly well
With delicate spire and whorl,
How exquisitely minute,
A miracle of design!
What is it? a learned man
Could give it a clumsy name.
Let him name it who can;
The beauty will be the same.

– Alfred, Lord Tennyson, *Maud* II, II.

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Preface

A substantial part of the material covered in these notes formed the content of a course of lectures given during the Spring of 1985 in the Mathematics Institute of the University of Warwick, England. My aim was to introduce the aspiring graduate student to a beautiful and central part of mathematics, the representation theory of semisimple groups. This is of course a vast and active subject, bringing together at a fairly deep level algebra, geometry, analysis, and arithmetic. This is one of the reasons why it is difficult to get into, at least for the young student. I therefore made an attempt to keep the requirements minimal, and introduced the major themes of the subject by first working them out in the case of $SL(2, \mathbb{R})$. This approach has no claim to novelty: it has been done before, and there is the well-known book of S. Lang dealing only with $SL(2, \mathbb{R})$. I have, however, discussed a number of topics not treated by Lang, such as the Schwartz space, invariant eigendistributions, wave packets, and so on; in addition, I have included, wherever possible, indications of how these ideas may be generalized to the context of a general semisimple Lie group.

The organization of the book is not always linear because I wanted to adhere closely to the lectures and preserve their freeflowing nature. As a result, the reader will often find references to matters that are not defined or are quite advanced, especially in Chapters 1–2. This should not discourage him (or her); my advice to the reader is to ignore them and proceed ahead, and come back to the difficult points later. Chapters 4–8 are essentially linear and can be worked through by a graduate student (late first year or early second year in an American university). I have included appendices on Functional Analysis and Lie theory that offer the reader some basic definitions, explanations of some concepts, and some historical perspective.

It is a great pleasure for me to express my gratitude to the Mathematics Institute of the University of Warwick and the Science and Engineering Research Council of the United Kingdom for inviting me to Warwick, and to the staff and faculty of the Institute for their wonderful hospitality. To thank Klaus and Annelise Schmidt adequately for what they did for my wife and me is impossible; it was only through their kindness and generosity that this visit become so memorable for us. Finally I want to thank the

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Cambridge University Press and their editor Dr David Tranah for being really patient while I took my time preparing the final version of this manuscript.

Pacific Palisades, October 1986

V.S. Varadarajan