1

Introduction

1.1 Aim

The aim of this book is to introduce advanced undergraduates or beginning graduate students to the subject of harmonic analysis on semisimple Lie groups. This involves doing a certain amount of representation theory for these groups, either implicitly or explicitly, because harmonic analysis is concerned mainly with expanding arbitrary functions (and generalized functions) on a group as a series or integral of functions which occur as matrix elements of irreducible representations of the group. Nevertheless this is not a book on representation theory. As long as the group is compact, the harmonic analysis point of view is not very prominent; but for noncompact groups the behaviour of the matrix elements at infinity becomes critical, and the analysis becomes decisive. Thus, although representation theory and harmonic analysis have a lot in common, the two subjects are not quite the same; and the differences will become clear to the reader when both themes have been developed to a certain extent. In this introductory chapter I shall discuss briefly a number of sources of motivation for studying representations and harmonic analysis, whose diversity and wide-ranging nature show that our subject is much more central than it seems at first sight.

1.2 Some definitions

A *representation* of a group G is a homomorphism of G into the general linear group $GL(V)$ of a complex finite-dimensional vector space V; the representation is said to be *in* (or *on*) V. If 0 and V are the only linear subspaces of V stable under the representation π (i.e., left invariant by $\pi(g)$ for all $g \in G$), then π is said to be *irreducible*. Representations π_j in V_j ($j = 1, 2$) are *equivalent* if there is a linear isomorphism $T(V_1 \to V_2)$ such that $\pi_2(g) = T\pi_1(g)T^{-1}$ for all $g \in G$. If π_j are representations of G on V_j ($j = 1, \ldots, m$) their direct sum $\pi = \pi_1 \oplus \cdots \oplus \pi_m$ and tensor product $\pi' = \pi_1 \otimes \cdots \otimes \pi_m$ are the representations defined respectively in $V = V_1 \oplus \cdots \oplus V_m$ and $V' = V_1 \otimes \cdots \otimes V_m$ by

$$\left.\begin{aligned} \pi(g) &= \pi_1(g) \oplus \cdots \oplus \pi_m(g) \\ \pi'(g) &= \pi_1(g) \otimes \cdots \otimes \pi_m(g) \end{aligned}\right\} \quad (g \in G)$$

The dual (or contragradient) of a representation π in V is the representation

π^* in the dual V^* of V defined by

$$(v, \pi^*(g)v^*) = (\pi(g^{-1})v, v^*) \quad (v \in V, v^* \in V^*, g \in G)$$

Actually, the category of finite-dimensional representations, with $\oplus$, $\otimes$, $*$ defined as above, is adequate only for problems involving finite groups. For infinite groups, it is necessary to impose additional restrictions such as continuity, rationality, and so on, as well as to consider infinite-dimensional representations. We begin by looking at some of the most common examples.

Finite groups. As we remarked above, the category of representations in finite-dimensional vector spaces is the natural one to work with, but the restriction to complex vector spaces is not reasonable. Most applications, in physics and chemistry, for example, deal with complex representations; but for a general theory the underlying fields should be arbitrary [S1].

Algebraic groups. In concrete terms these are groups of matrices defined by polynomial conditions on their entries. Typical examples are the unimodular group, i.e., the group of matrices of determinant 1, the orthogonal group, the symplectic group, and so on. For such a group defined over (i.e., with entries from) an algebraically closed field k, it is natural to work only with finite-dimensional representations π which are *rational*; here rational means the entries of $\pi(g)$ relative to a basis of V are polynomial functions of the entries of g and $\det(g)^{-1}$.

Topological groups. Impulses from functional analysis and quantum physics were very much responsible for a systematic study of representations in infinite-dimensional spaces. The groups considered are topological and the vector spaces usually complete and locally convex. If G is a topological group and V is a complete locally convex space, the homomorphism π of G into the group of invertible automorphisms of V is a *representation* if the map $(g, v) \mapsto \pi(g)(v)$ of $G \times V$ into V is continuous. Important special cases are when V is a Banach space and G is locally compact. If V is a Hilbert space and each $\pi(g)$ is unitary, π is called a *unitary representation*. For a general π, irreducibility now means that 0 and V are the only *closed* linear subspaces stable under π; equivalence of π_1 and π_2 is defined as before, but with T required now to be a topological linear isomorphism; if π_1 and π_2 are unitary, and T is unitary, π_1 and π_2 are said to be *unitary equivalent*. The set of equivalence classes of irreducible unitary representations of G is written $\hat{G}$ and is called the *unitary dual* of G. One can define infinite (orthogonal) direct sums in the category of unitary representations. The definition of tensor products for infinite-dimensional representations is somewhat technically involved since tensor product is a technically complicated notion for topological vector spaces; we shall not make any serious use of this.

1.3 Classical invariant theory

The geometric invariant theory of classical geometers was one of the first examples of an important context where representation theory entered in a nontrivial way. Here $G = SL(n+1, \mathbb{C})$ and one starts with a rational representation of the algebraic group G in a complex vector space V. The action of G on V gives rise to an action on the projective space $\mathbb{P}(V)$ of V. The problem of invariant theory is that of describing the orbit space $G\backslash\mathbb{P}(V)$[MF]. This leads almost immediately to the study of the action of G on the rings of functions on $\mathbb{P}(V)$. Let R be the graded ring of polynomials on V, and $\bar{R} = R^G$ be the graded subring of G-invariant polynomials. The first step in the description of $G\backslash\mathbb{P}(V)$ is the study of the following question:

$$\text{Is } \bar{R} \text{ finitely generated?} \tag{$*$}$$

Hilbert proved, at the beginning of his epoch-making work on invariant theory, that for $G = SL(n+1, \mathbb{C})$ the answer to $(*)$ is affirmative. This was eventually extended to all complex semisimple groups G by Hermann Weyl who obtained it as a consequence of his famous theorem that all rational representations of any complex semisimple group are completely reducible, i.e., direct sums of irreducible representations. Weyl's theorem is one of the deepest and most important in the finite-dimensional representation theory of semisimple groups, and we shall discuss it briefly in the next chapter.

When the questions of geometric invariant theory were examined by Mumford in the 1960s Chevalley had already developed the theory of semisimple algebraic groups over any algebraically closed field k; and Mumford's investigations led naturally to the question of finite generation of $\bar{R} = R^G$ where R is the graded ring of polynomials on a vector space V on which we have a rational representation of the algebraic semisimple group G. Unfortunately Weyl's method fails when $\operatorname{char}(k) > 0$; representations of G are not in general completely reducible when k has positive characteristic. Nevertheless Mumford conjectured that all rational representations of the semisimple group G over an arbitrary algebraically closed field k possess the following property (M):

> if $v \neq 0$ is a vector in the space V of the given representation and v is invariant under G, there is a nonzero homogeneous polynomial f on V invariant under G such that $f(v) \neq 0$.

The property (M) is equivalent to complete irreducibility when $\operatorname{char}(k) = 0$;

its validity for general k implies $\bar{R}$ is finitely generated. Mumford's conjecture was proved by Haboush in 1975 [Hb]. These results have been the beginning of new progress in representation theory and geometric invariant theory [MF]. For the classical theory there are of course many references; in addition to Weyl's great classic [W1] the reader may consult Schur's lectures [Sc].

1.4 Quantum mechanics and unitary representations

We now turn to a completely different source of problems in which unitary representations appear prominently. The group G is now the symmetry group of a quantum-mechanical system and one is interested in a description of the system that is covariant under G. Now, any quantum-mechanical description requires the introduction of a complex Hilbert space $\mathscr{H}$; the physical interpretation consists in identifying the ortho-complemented lattice $\mathscr{L}(\mathscr{H})$ of the closed linear subspaces of $\mathscr{H}$ with the logic of experimentally verifiable propositions of the system [V1]. The requirement of covariance means there is a homomorphism σ of G into the group of automorphisms of $\mathscr{L}(\mathscr{H})$. Now it can be proved that any automorphism σ of $\mathscr{L}(\mathscr{H})$ is induced in the obvious way by a unitary or antiunitary operator, determined uniquely up to a multiplicative constant of absolute value 1. Under mild assumptions on G and σ it can be shown that σ is induced by a *projective unitary representation* of G, i.e. a unitary representation of *an extension of* G *by the group* T *of complex numbers of absolute value* 1. This representation is obviously an important invariant of the system. For suitable G one can show that σ is induced by a unitary representation of its simply connected covering group $\tilde{G}$. This is the case when G is the group of automorphisms of Euclidean or Minkowskian affine space-time (however, this is *not* the case for the group of automorphisms of Galilean space-time).

If G is the group of automorphisms of an affine space with the structure of Minkowskian space-time, G can be written as a semidirect product $A \ltimes H$ where A is the four-dimensional group of space-time translations, $H = SL(2, \mathbb{C})$, and H acts linearly on A via the Lorentz transformations. In any description of a quantum-mechanical system consistent with special relativity there will thus appear a unitary representation of G. For instance, if the system is that of a free elementary particle, it is natural to expect this representation to be irreducible, and to expect further that it will tell us everything about this free particle. Thus the free relativistic elementary particles are in one-one correspondence with a certain subset of $\hat{G}$. Now there is a general method, due to Mackey, for determining the irreducible unitary representations of such (and even more general) locally compact semidirect products. This method, applied in the present situation, leads in a simple and natural manner to the classification of the particles in terms of their mass and spin [V1] [Ma1].

It is not always the case that the symmetry groups are locally compact. The *gauge groups* occurring in the theory of gauge fields are infinite-dimensional, and the representation theory of these and more general groups is quite active now, although not yet in any definitive state [K] [PS].

1.5 Classical Fourier analysis. Plancherel and Poisson formulae

The starting point of Fourier analysis is the idea that a more or less arbitrary function can be expanded as a 'linear combination' of the exponentials. The basic objective of the theory is to define the *Fourier transform*; the transform of a function (or a generalized function) shows how it is made up of its harmonic constituents. We shall now explain briefly the point of view of the theory of unitary representations that allows us to understand and generalize these classical themes.

Fourier series deal with functions on the torus $\mathbb{T}^n$ with coordinates $\theta = (\theta_1, \ldots, \theta_n)$. We introduce the Hilbert space $L^2(\mathbb{T}^n) = L^2(\mathbb{T}^n, d\theta)$, $d\theta = d\theta_1 \cdots d\theta_n$, $\int d\theta = 1$. For $\phi \in \mathbb{T}^n$ we define the linear operator $\lambda(\phi)$ on $L^2(\mathbb{T}^n)$ by

$$(\lambda(\phi)f)(\theta) = f(-\phi + \theta)$$

The $\lambda(\phi)$ are unitary, and it is easy to show that $\lambda(\phi \to \lambda(\phi))$ is a unitary representation of $\mathbb{T}^n$, the so-called *regular representation* of $\mathbb{T}^n$. The *irreducible* unitary representations of $\mathbb{T}^n$ are precisely all the *characters*

$$\chi_m : \theta \to \exp 2\pi i(m_1\theta_1 + \cdots + m_n\theta_n) \quad (m = (m_1, \ldots, m_n) \in \mathbb{Z}^n)$$

The functions χ_m are in $L^2(\mathbb{T}^n)$; the one-dimensional subspaces $\mathbb{C}\cdot\chi_m$ are stable under λ, and the restriction of λ to $\mathbb{C}\cdot\chi_{-m}$ is equivalent to the one-dimensional representation χ_m. The orthogonal direct sum decomposition

$$L^2(\mathbb{T}^n) = \bigoplus_m \mathbb{C}\cdot\chi_{-m}$$

shows that λ is equivalent to the infinite (orthogonal) direct sum of the χ_m ($m \in \mathbb{Z}^n$), each taken only once. We shall now see that the Fourier transform operator leads to the explicit 'diagonalization' of λ. For any $f \in L^2(\mathbb{T}^n)$ define its Fourier transform $\hat{f} = \mathscr{F} f$ by

$$\hat{f}(m) = (f, \chi_{-m}) \quad (m \in \mathbb{Z}^n)$$

where $(\cdot, \cdot)$ is the scalar product. Then $\hat{f}$ is a function on $\mathbb{Z}^n$. If we equip $\mathbb{Z}^n$ with the counting measure and introduce the Hilbert space $L^2(\mathbb{Z}^n)$, then $\hat{f} \in L^2(\mathbb{Z}^n)$ and

$$\mathscr{F} : f \mapsto \hat{f}$$

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is a *unitary isomorphism* of $L^2(\mathbb{T}^n)$ with $L^2(\mathbb{Z}^n)$:

$$\|f\| = \|\hat{f}\|$$

which is the usual Parseval relation. The inverse operator $\mathscr{F}^{-1}$ is given by

$$f = \sum_m \hat{f}(m)\chi_{-m}$$

the series converging in $L^2(\mathbb{T}^n)$. If we now use $\mathscr{F}$ to carry the representation λ to a representation μ of $\mathbb{T}^n$ in $L^2(\mathbb{Z}^n)$, $\mu = \mathscr{F}\circ\lambda\circ\mathscr{F}^{-1}$, then

$$\mu(\phi)\hat{f}(m) = \chi_m(\phi)\hat{f}(m) \quad (\phi \in \mathbb{T}^n)$$

This formula shows that in the standard basis of $L^2(\mathbb{Z}^n)$ all the operators $\mu(\phi)$ are diagonal.

If f is smooth, $\hat{f}(m)$ tends to 0 very rapidly when $|m| \to \infty$; the series

$$f = \sum_m \hat{f}(m)\chi_{-m}$$

then converges very nicely: we have, for it as well as for all the series obtained by formal differentiation, uniform convergence. In particular,

$$f(0) = \sum_m \hat{f}(m) \quad (f \in C^\infty(\mathbb{T}^n)) \tag{P}$$

We shall refer to this as the *Plancherel formula*.

For $\mathbb{R}^n$ the theory is more delicate. The characters of $\mathbb{R}^n$ are the functions

$$\chi_t : (x_1, \ldots, x_n) \to \exp i(t_1x_1 + \cdots + t_nx_n) \quad (t \in \mathbb{R}^n)$$

The regular representation of $\mathbb{R}^n$ is defined as before; it acts on $L^2(\mathbb{R}^n)$ by

$$(\lambda(y)f)(x) = f(-y + x)$$

Proceeding as before we define the *Fourier transform* of f by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x)\chi_t(x)\,dx \quad (t \in \mathbb{R}^n) \tag{FT}$$

Then

$$(\lambda(y)f)^\wedge(t) = \chi_t(y)\hat{f}(t) \quad (y, t \in \mathbb{R}^n) \tag{M}$$

so that the operators $\lambda(y)$ become multiplication operators simultaneously, and thus are 'diagonalized'. However, the χ_t are of absolute value 1 and so do not lie in the Hilbert space, so that the definition of the Fourier transform in (FT) is not strictly valid for all f in $L^2(\mathbb{R}^n)$. The traditional way to overcome this difficulty is to use the definition (FT) initially for f suitably restricted, say for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$; the key step is then to prove that on this restricted domain the map $f \mapsto \hat{f}$ is essentially unitary, and then to complete its definition to all of $L^2(\mathbb{R}^n)$ by continuity, noting that $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. The restricted unitarity is proved in the form

$$\int_{\mathbb{R}^n} |f(x)|^2 \,\mathrm{d}x = (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{f}(t)|^2 \,\mathrm{d}t \qquad (\mathrm{P}_1)$$

It then turns out that the Fourier transform maps $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$ so that it is a unitary isomorphism:

$$\mathscr{F}: L^2(\mathbb{R}^n, \mathrm{d}x) \cong L^2(\mathbb{R}^n, (2\pi)^{-n}\,\mathrm{d}t)$$

The relations (M) are now valid rigorously and show that the representation $\mu = \mathscr{F} \circ \lambda \circ \mathscr{F}^{-1}$ acts by multiplication operators. The formula (P_1), valid for all $f \in L^2(\mathbb{R}^n)$, is the *Plancherel formula.*

The 'diagonalization' of λ effected by $\mathscr{F}$ is the classic example of a 'continuous decomposition'. Let us introduce an equivalence relation in the σ-algebra of Borel subsets of $\mathbb{R}^n$ by defining $E \sim F$ to mean $(E\backslash F) \cup (F\backslash E)$ has measure zero. If $\mathscr{B}$ is the set of equivalence classes, $\mathscr{B}$ is a σ-algebra also; but unlike the σ-algebra of Borel sets $\mathscr{B}$ has no atoms. Further, Lebesgue measure becomes a measure on $\mathscr{B}$ with the property that each nonzero element has measure strictly greater than zero. For any Borel set E let

$$S(E) = \{f \mid f \in L^2(\mathbb{R}^n, \mathrm{d}x)\}, \hat{f} = 0 \text{ outside } E\}$$

It is then easy to show using (M) that $S(E)$ is a closed λ-stable subspace of $L^2(\mathbb{R}^n, \mathrm{d}x)$. Of course $S(E)$ depends only on the equivalence class of E and so we have a map $S(e \mapsto S(e))$ from the σ-algebra $\mathscr{B}$ to the orthocomplemented lattice Λ of the λ-stable closed linear subspaces of $L^2(\mathbb{R}^n, \mathrm{d}x)$. It is not difficult to show at this stage that S is an *isomorphism*:

$$S: \mathscr{B} \cong \Lambda$$

The most elegant way to prove all of these assertions is by using the Schwartz space; this method will also bring out the duality explicitly, and will have the additional advantage of focussing on the differential aspects of the theory. The Schwartz space of $\mathbb{R}^n$ is the space $\mathscr{S}$ of all C^∞ functions f on $\mathbb{R}^n$ such that for any integers $m \geqslant 0, \alpha_1, \ldots, \alpha_n \geqslant 0$,

$$((\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} f)(x) = \mathrm{O}((1 + x_1^2 + \cdots + x_n^2)^{-m})$$

when $x_1^2 + \cdots + x_n^2 \to \infty$. If we introduce the seminorms

$$\mu_{\alpha,m}(f) = \sup |(D^\alpha f)(x)|(1 + x_1^2 + \cdots + x_n^2)^m$$

$(D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n})$, we can view $\mathscr{S}$ as a topological vector space also. It is easy to show that $\mathscr{S}$ is complete (so that $\mathscr{S}$ is a Fréchet space). The differential operators D^α act as continuous linear operators on $\mathscr{S}$. Also, if we write, for any smooth function g, M_g for the operator of multiplication by g, then, under suitable assumptions on g, M_g will be a continuous linear operator on $\mathscr{S}$. For instance this is true if g is a polynomial. More generally, if g is of moderate growth in the sense that for any α there is an integer $m(\alpha) \geqslant 0$ and a constant $C(\alpha) > 0$ such that

$$|(D^\alpha g)(x)| \leqslant C(\alpha)(1 + x_1^2 + \cdots + x_n^2)^{m(\alpha)}$$

for all $x \in \mathbb{R}^n$, then $M_g(f \mapsto gf)$ is a well-defined and continuous operator of $\mathscr{S}$.

The rapid decay of the elements of $\mathscr{S}$ at infinity means that $\mathscr{S} \subset L^1(\mathbb{R}^n)$ and shows at once that the Fourier transform is defined by (FT) for all $f \in \mathscr{S}$; moreover, $\hat{f}$ will be a smooth function of t, and we have the formula

$$(-\mathrm{i}\partial/\partial t_1)^{\beta_1} \cdots (-\mathrm{i}\partial/\partial t_n)^{\beta_n} \hat{f} = (M_{x_1}^{\beta_1} \cdots M_{x_n}^{\beta_n} f)^\wedge$$

(x_j is the coordinate function $(x_1, \ldots, x_n) \mapsto x_j$). Furthermore, replacing f by its derivatives and integrating by parts we find the dual formula

$$((\mathrm{i}\partial/\partial x_1)^{\alpha_1} \cdots (\mathrm{i}\partial/\partial x_n)^{\alpha_n} f)^\wedge = M_{t_1}^{\alpha_1} \cdots M_{t_n}^{\alpha_n} \hat{f}$$

The estimate

$$|\hat{g}(t)| \leqslant \|g\|_1 \quad (g \in \mathscr{S})$$

in conjunction with the above formula now shows that $\hat{f} \in \mathscr{S}$ and that

$$\mathscr{F} : f \mapsto \hat{f}$$

is a continuous linear map of $\mathscr{S}$ into $\mathscr{S}$. The basic theorem may now be formulated as the assertion that $\mathscr{F}$ is a topological linear isomorphism of $\mathscr{S}$ with itself, and that $\mathscr{F}^{-1}$ is given by the inversion formula

$$(\mathscr{F}^{-1}g)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} g(t)\chi_t(x)\,\mathrm{d}t \qquad \text{(INV)}$$

If we take $x = 0$ and $g = \hat{f}$ in (INV) we get the Plancherel formula in the form

$$f(0) = (2\pi)^{-n} \int_{\mathbf{R}^n} \hat{f}(t)\,\mathrm{d}t \quad (f \in \mathscr{S}) \qquad (\mathrm{P}_2)$$

To get the earlier version (P_1) it suffices to take $f = g * \tilde{g}$ where $\tilde{g}(x) = \overline{g(-x)}$, and $*$ is convolution, defined by

$$(h_1 * h_2)(x) = \int_{\mathbf{R}^n} h_1(y)h_2(-y + x)\,\mathrm{d}y \quad (x \in \mathbb{R}^n)$$

for $h_1, h_2 \in \mathscr{S}$; then

$$\hat{f} = |\hat{g}|^2, f(0) = \int_{\mathbf{R}^n} |g(x)|^2\,\mathrm{d}x,$$

and so (P_1) follows from (P_2).

The Plancherel formula (P_2) has an interpretation from the standpoint of Schwartz's theory of distributions [Sch] which is important for us. Let us recall that a tempered distribution on $\mathbb{R}^n$ is nothing other than a continuous linear functional on $\mathscr{S}$. If F is a measurable function such that for some constants $c > 0$, $m \geqslant 0$ we have, for all $x \in \mathbb{R}^n$,

$$|F(x)| \leqslant c(1 + x_1^2 + \cdots + x_n^2)^m$$

then

$$f \mapsto \int F(x) f(x)\, dx$$

is a tempered distribution, which is usually identified with F. If δ is the Dirac measure at the origin,

$$\delta(f) = f(0) \quad (f \in \mathscr{S})$$

then (P_2) may be written as

$$\delta = (2\pi)^{-n} \int \chi_t\, dt \qquad (P_3)$$

interpreted as an identity of tempered distributions. It is also common to refer to (P_3) as the Plancherel formula.

The relation (P_3) is a special case of more general formulae that express how an arbitrary tempered distribution can be written as a 'linear combination' of the χ_t. Since $\mathscr{F}$ is an isomorphism of $\mathscr{S}$ with itself, it defines an isomorphism of the dual of $\mathscr{S}$ with itself. Thus, if T is a tempered distribution on $\mathbb{R}^n$, its *Fourier transform* $\hat{T}$ is the tempered distribution defined by

$$\hat{T}(f) = T(\hat{f}) \quad (f \in \mathscr{S}) \qquad (FT_1)$$

The map $T \mapsto \hat{T}$ is a linear isomorphism of the space of tempered distributions with itself.

If $T = t\, dx$ in the sense that $T(f) = \int t f\, dx$, t being an element of $\mathscr{S}$, it is immediate that $\hat{T} = \hat{t}\, dt$, so that (FT_1) is consistent with the earlier definition of Fourier transform on $\mathscr{S}$. In the general case it is usual to write (FT_1) in the symbolic form

$$T(x) = (2\pi)^{-n} \int \hat{T}(t) \chi_t(x)\, dt$$

that shows how T is resolved into its harmonic constituents.

A number of classical formulae may be regarded as the computation of $\hat{T}$ for suitable T. The example that is most interesting from the point of view of arithmetic is the case when T is the counting measure on $\mathbb{Z}^n$:

$$T = \sum_{m \in \mathbb{Z}^n} \delta_m$$

Here δ_a is the Dirac measure at a, $\delta_a(f) = f(a)$. It is easy to show that T is tempered. The *Poisson* (*summation*) *formula* is the statement

$$\hat{T} = \sum_{m \in \mathbb{Z}^n} \delta_{2\pi m} \qquad (\mathrm{Po}_1)$$

which is often written as

$$\sum_{m \in \mathbb{Z}^n} f(m) = \sum_{q \in 2\pi\mathbb{Z}^n} \hat{f}(q) \quad (f \in \mathscr{S}) \qquad (\mathrm{Po}_2)$$

If we view $\mathbb{R}^n/\mathbb{Z}^n$ as the torus $\mathbb{T}^n$, then we have a unitary representation of $\mathbb{R}^n$ in $L^2(\mathbb{T}^n)$ induced by the action of $\mathbb{R}^n$ on $\mathbb{T}^n$ by translations: if γ is the natural map $\mathbb{R}^n \to \mathbb{T}^n$, the unitary representation in question, say π, is defined by

$$(\pi(y)f)(\gamma(x)) = f(\gamma(-y+x)) \quad (x, y \in \mathbb{R}^n)$$

for $f \in L^2(\mathbb{T}^n)$. The Poisson formula is entirely equivalent to the statement that

$$\pi = \bigoplus_{q \in 2\pi\mathbb{Z}^n} \chi_q$$

1.6 Fourier analysis on abelian groups

The work of Pontryagin and van Kampen in the 1920s and 1930s led to a detailed understanding of the structure of locally compact abelian groups. It then became possible to develop harmonic analysis on these groups and place the classical theory in a proper perspective. The classic work treating this is that of A. Weil [We1]. The perspective of abelian Fourier analysis would eventually prove to be the starting point of a number of related developments such as Gel'fand's theory of commutative Banach algebras, the Artin–Tate extension of Hecke's work on L-functions and their functional equations (about this we shall say more in §7), Mackey's work on unitary representations of semidirect products, the work of Weil himself [We3] that revealed the deep-lying relationship of unitary representation theory to number theory and the theory of quadratic forms, and so on.

Let G be a locally compact abelian group which we shall assume for simplicity to be second-countable. We denote by $\hat{G}$ the set of *characters* of G, i.e., the set of all continuous homomorphisms of G into T, the multiplicative group of complex numbers of absolute value 1. It follows from the spectral theorem for unitary operators that the characters of G are precisely the irreducible unitary representations of G so that $\hat{G}$ is nothing else than the unitary dual of G in this case. Under pointwise multiplication $\hat{G}$ becomes a group, and we equip it with the topology of uniform convergence on compact subsets of G. It is then a fundamental fact that $\hat{G}$ is a locally compact abelian second-countable group also. For $x \in G$, $\hat{x} \in \hat{G}$, we write

$$\langle x, \hat{x} \rangle = \hat{x}(x)$$