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Classical Groups as Linear Algebraic Groups

Hermann Weyl, in his famous book (Weyl [1946]), gave the name *Classical Groups* to certain families of groups of linear transformations. In this chapter we introduce these groups and study them from the point of view of linear algebraic groups. We develop the theory of algebraic groups over $\mathbb{C}$ far enough to obtain the basic results on their representations, Lie algebras, and Jordan–Chevalley decompositions. We introduce the notion of a *real form* of an algebraic group (considered as a Lie group) and we describe the compact real forms of the classical groups.

1.1 Linear Algebraic Groups

1.1.1 Definitions and Examples

Let $\mathrm{GL}(n, \mathbb{C})$ be the group of invertible $n \times n$ complex matrices, and let $M_n(\mathbb{C})$ be the space of all $n \times n$ complex matrices. For $y \in M_n(\mathbb{C})$ and $1 \leq i, j \leq n$ we write $x_{ij}(y)$ for the i, j entry in y. A complex-valued function f on $M_n(\mathbb{C})$ is a *polynomial function* if

$$f(y) = p(x_{11}(y), x_{12}(y), \ldots, x_{nn}(y)),$$

where $p \in \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$. A subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ is a *linear algebraic group* if there is a set A of polynomial functions on $M_n(\mathbb{C})$ so that

$$G = \{g \in \mathrm{GL}(n, \mathbb{C}) : f(g) = 0 \text{ for all } f \in A\}.$$

Examples

The *Hilbert basis theorem* (Theorem A.1.1) asserts that any algebraic group can be defined by a *finite* number of polynomial equations. We now give some examples for which this property obviously holds.

1. The basic example of a linear algebraic group is the general linear group $\mathrm{GL}(n, \mathbb{C})$ (take the defining set A of relations to consist of the 0 polynomial). In the case $n = 1$ we have $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$, the multiplicative group of the field $\mathbb{C}$.

2. Let $D_n \subset \mathrm{GL}(n, \mathbb{C})$ be the subgroup of diagonal matrices. The defining equations for D_n are $x_{ij}(g) = 0$ for all $i \neq j$, so D_n is an algebraic group.
3. Let $N_n \subset \mathrm{GL}(n, \mathbb{C})$ be the subgroup of upper-triangular matrices with diagonal entries 1. The defining equations in this case are $x_{ii}(g) = 1$ for all i and $x_{ij}(g) = 0$ for all $i > j$. When $n = 2$, the group N_2 is isomorphic (as an abstract group) to the additive group of the field $\mathbb{C}$, via the map

$$z \mapsto \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

from $\mathbb{C}$ to N_2. We will look upon $\mathbb{C}$ as the linear algebraic group N_2.
4. Let $B_n \subset \mathrm{GL}(n, \mathbb{C})$ be the subgroup of upper-triangular matrices. The defining equations for B_n are $x_{ij}(g) = 0$ for all $i > j$, so B_n is an algebraic group.

We now introduce four families of linear algebraic groups that will be our chief concern throughout the rest of the book. (In Weyl's words, "each group stands in its own right and does not deserve to be looked upon merely as a subgroup of something else, be it even Her All-Embracing Majesty GL(n).") A group G in a particular family is specified by an integer l, which is the dimension of the subgroup of diagonal matrices in G, relative to a suitably chosen basis for $\mathbb{C}^n$ (we shall verify later that l is intrinsically defined, independently of the matrix form of G). Each family is defined by a geometric invariance property.

Notation. We denote the identity matrix (of any size) by I. We denote the matrix transpose of $X \in M_n(\mathbb{C})$ as X^t. A matrix X is *symmetric* if $X^t = X$, and *skew symmetric* if $X^t = -X$. We view elements of $\mathbb{C}^n$ as column vectors ($n \times 1$ matrices) with $M_n(\mathbb{C})$ acting on the left, and the dual space $(\mathbb{C}^n)^*$ as row vectors ($1 \times n$ matrices).

Special Linear Group

The *special linear group* $\mathrm{SL}(n, \mathbb{C})$ consists of all matrices $g \in \mathrm{GL}(n, \mathbb{C})$ with $\det(g) = 1$. Since $\det(g)$ is a polynomial in the matrix entries of g, $\mathrm{SL}(n, \mathbb{C})$ is an algebraic group. We shall call it a group of *type* A_l, where $l = n - 1$. To see the geometric significance of this group, we recall that the nth exterior power $\bigwedge^n \mathbb{C}^n$ is one dimensional (an element in this space is a *complex volume form*). The group $\mathrm{GL}(n, \mathbb{C})$ acts on volume forms by multiplication by $\det(g)$, so $\mathrm{SL}(n, \mathbb{C})$ is the group of the linear transformations that preserve volume forms.

The other classical groups are defined by bilinear forms B on $\mathbb{C}^n$ (see Section B.2.1). If B is a bilinear form and $g \in \mathrm{GL}(n, \mathbb{C})$, define a bilinear form $g \cdot B$ by

$$g \cdot B(x, y) = B(g^{-1}x, g^{-1}y).$$

If B has matrix T then the bilinear form $g \cdot B$ has matrix $(g^t)^{-1} T g^{-1}$. If B is symmetric (resp. skew-symmetric), then so is $g \cdot B$. We say that B is *invariant* under g if $g \cdot B = B$. This is equivalent to

$$T = g^t T g. \tag{1.1.1}$$

Orthogonal Groups

Let B be a nondegenerate symmetric bilinear form on $\mathbb{C}^n$. The *orthogonal group* relative to B is

$$\mathrm{O}(\mathbb{C}^n, B) = \{g \in \mathrm{GL}(n, \mathbb{C}) : B(gx, gy) = B(x, y) \text{ for } x, y \in \mathbb{C}^n\}.$$

To characterize this group in matrix terms, let S be the matrix of the form: $B(x, y) = x^t S y$. Then S is a symmetric, invertible matrix. From (1.1.1) we have

$$g \in \mathrm{O}(\mathbb{C}^n, B) \Longleftrightarrow g^t S g = S. \tag{1.1.2}$$

This shows that $\mathrm{O}(\mathbb{C}^n, B)$ is an algebraic group, defined by quadratic relations on the matrix entries of g. It also suggests another description of $\mathrm{O}(\mathbb{C}^n, B)$. Define

$$\sigma_S(g) = S^{-1}(g^t)^{-1}S \qquad \text{for } g \in \mathrm{GL}(n, \mathbb{C}).$$

Then $\sigma_S(gh) = \sigma_S(g)\sigma_S(h)$ for $g, h \in \mathrm{GL}(n, \mathbb{C})$, $\sigma_S(I) = I$, and $\sigma_S(\sigma_S(g)) = g$ (such a map σ_S is called an *involutory automorphism* of $\mathrm{GL}(n, \mathbb{C})$). The group $\mathrm{O}(\mathbb{C}^n, B)$ is the set of *fixed points* of σ_S:

$$g \in \mathrm{O}(\mathbb{C}^n, B) \Longleftrightarrow \sigma_S(g) = g.$$

We say that a set of vectors $\{v_1, \ldots, v_n\} \subset \mathbb{C}^n$ is a *B-orthonormal basis* for $\mathbb{C}^n$ if $B(v_i, v_j) = \delta_{ij}$. When $B(x, y) = x^t y$, then the standard basis $e_1, \ldots, e_n$ for $\mathbb{C}^n$ is B-orthonormal. Hence $g \cdot B = B$ in this case precisely when the columns $g_i = ge_i$ of g form a B-orthonormal basis for $\mathbb{C}^n$.

Lemma 1.1.1 *Suppose $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are B-orthonormal bases for $\mathbb{C}^n$. Let $g \in \mathrm{GL}(n, \mathbb{C})$ be defined by $gv_i = w_i$ for $i = 1, \ldots, n$. Then $g \in \mathrm{O}(\mathbb{C}^n, B)$.*

Proof. We have $B(gv_i, gv_j) = B(w_i, w_j) = B(v_i, v_j)$ for all i, j since both bases are B-orthonormal. Hence $B(gx, gy) = B(x, y)$ for all $x, y \in \mathbb{C}^n$. ■

Let $g \in \mathrm{O}(\mathbb{C}^n, B)$. Since $\det(S) = \det(g^t S g) = \det(g)^2 \det S$, we see that $\det g = \pm 1$. We define

$$\mathrm{SO}(\mathbb{C}^n, B) = \{g \in \mathrm{O}(\mathbb{C}^n, B) : \det g = 1\}$$

and call it the *special orthogonal group* relative to B.

Proposition 1.1.2 *Let B, B' be nondegenerate symmetric bilinear forms on $\mathbb{C}^n$. Then there exists $\gamma \in \mathrm{GL}(n, \mathbb{C})$ such that $\mathrm{O}(\mathbb{C}^n, B') = \gamma \mathrm{O}(\mathbb{C}^n, B)\gamma^{-1}$.*

Proof. By Lemma 1.1.1 there exists a B-orthonormal basis $v_1, \ldots, v_n$ and a B'-orthonormal basis $v_1', \ldots, v_n'$ for $\mathbb{C}^n$. Let $\gamma \in \mathrm{GL}(n, \mathbb{C})$ be defined by $\gamma v_i = v_i'$. If S, S' are the matrices defining the forms B, B', then

$$\delta_{ij} = (S'\gamma v_i, \gamma v_j) = (\gamma^t S' \gamma v_i, v_j) = (S v_i, v_j).$$

Hence $\gamma^t S' \gamma = S$. Let $g \in \mathrm{GL}(n, \mathbb{C})$ and set $h = \gamma g \gamma^{-1}$. Then

$$h^t S' h = (\gamma^{-1})^t g^t \gamma^t S' \gamma g \gamma^{-1} = (\gamma^{-1})^t g^t S g \gamma^{-1}.$$

Thus Equation (1.1.2) implies that $g \in \mathrm{O}(\mathbb{C}^n, B)$ if and only if $h \in \mathrm{O}(\mathbb{C}^n, B')$. ■

A vector $v \in \mathbb{C}^n$ is *B-isotropic* if $B(v, v) = 0$. A subspace $W \subset \mathbb{C}^n$ is *B-isotropic* if $B(u, v) = 0$ for all $u, v \in W$, and it is *maximal isotropic* if there is no larger B-isotropic subspace containing it.

Maximal isotropic subspaces will play an important role when we examine the structure of the complex orthogonal groups in more detail. For example, let $n = 2$ and take the bilinear form B on $\mathbb{C}^2$ to have matrix

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The vectors e_1 and e_2 are B-isotropic. We calculate that

$$g^t S g = \begin{bmatrix} 2ac & ad+bc \\ ad+bc & 2bd \end{bmatrix} \qquad \text{for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

Since $ad - bc = 1$, it follows that $ad + bc = 2ad - 1$. Hence $g^t S g = S$ if and only if $ad = 1$ and $b = c = 0$. Thus $\mathrm{SO}(\mathbb{C}^2, B)$ consists of the matrices

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \qquad a \in \mathbb{C}^\times$$

and is isomorphic to $\mathrm{GL}(1, \mathbb{C})$.

When $n = 2l$, a basis $\{v_1, \ldots, v_l, v_{-1}, \ldots, v_{-l}\}$ for $\mathbb{C}^n$ that satisfies $B(v_i, v_j) = 0$ for $i \neq -j$ and $B(v_i, v_{-i}) = 1$ for all i, j will be called a *B-isotropic basis*. When $n = 2l+1$, a basis $\{v_0, v_1, \ldots, v_l, v_{-1}, \ldots, v_{-l}\}$ for $\mathbb{C}^n$ that satisfies $B(v_i, v_j) = 0$ for $i \neq -j$ and $B(v_i, v_{-i}) = 1$ for all $i, j = 0, \pm 1, \ldots, \pm l$ will be called a *B-isotropic basis*. Such a basis always exists (see Section B.2.1).

Lemma 1.1.3 *Suppose $\{u_i\}$ and $\{v_i\}$ are B-isotropic bases for $\mathbb{C}^n$. Define $g \in \mathrm{GL}(n, \mathbb{C})$ by $gu_i = v_i$ for all i. Then $g \in \mathrm{O}(\mathbb{C}^n, B)$.*

Proof. Just as in the proof of Lemma 1.1.1, we see that $B(gx, gy) = B(x, y)$ for all $x, y \in \mathbb{C}^n$ by verifying this relation on the given basis vectors. ■

We call $\mathrm{SO}(\mathbb{C}^{2l}, B)$ a group of *type* D_l and $\mathrm{SO}(\mathbb{C}^{2l+1}, B)$ a group of *type* B_l. The reasons for distinguishing between the even and odd cases will become clear later when we determine the structure of these groups. When B is a nondegenerate symmetric bilinear form on a finite-dimensional complex vector space V, we shall also write $\mathrm{O}(V, B)$ and $\mathrm{SO}(V, B)$ to refer to the corresponding orthogonal and special orthogonal groups. We shall denote these groups as $\mathrm{O}(V)$ and $\mathrm{SO}(V)$ when the particular choice of B is understood or irrelevant. When $V = \mathbb{C}^n$ we will generally use the notation $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$ for any element of the corresponding isomorphism class of linear algebraic groups.

The group $\mathrm{SO}(\mathbb{C}^n, B)$ has index two in $\mathrm{O}(\mathbb{C}^n, B)$. Indeed, when $n = 2l + 1$ is odd, then $-I$ is orthogonal with determinant -1, and

$$\mathrm{O}(\mathbb{C}^{2l+1}, B) = \mathrm{SO}(\mathbb{C}^{2l+1}, B) \cup (-I) \cdot \mathrm{SO}(\mathbb{C}^{2l+1}, B).$$

When $n = 2l$ is even, let $\{v_{\pm 1}, \ldots, v_{\pm l}\}$ be a B-isotropic basis for $\mathbb{C}^n$. The transformation ξ, which interchanges v_l and v_{-l} and fixes the other basis elements, is orthogonal (by Lemma 1.1.3) and has determinant -1. Thus

$$\mathrm{O}(\mathbb{C}^{2l}, B) = \mathrm{SO}(\mathbb{C}^{2l}, B) \cup \xi \cdot \mathrm{SO}(\mathbb{C}^{2l}, B).$$

Symplectic Groups

Let Ω be a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^n$. Then $n = 2l$ must be even. We define the *symplectic group* relative to Ω as

$$\mathrm{Sp}(\mathbb{C}^{2l}, \Omega) = \{g \in \mathrm{GL}(2l, \mathbb{C}) : \Omega(gx, gy) = \Omega(x, y) \text{ for } x, y \in \mathbb{C}^{2l}\}.$$

Let $J \in M_{2l}(\mathbb{C})$ be such that $\Omega(x, y) = x^t J y$. Then J is skew symmetric and nonsingular, and

$$g \in \mathrm{Sp}(\mathbb{C}^{2l}, \Omega) \Longleftrightarrow g^t J g = J. \tag{1.1.3}$$

Hence $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ is an algebraic group defined by quadratic relations on the matrix entries of g, just as in the case of the orthogonal groups. As in that case there is an associated involutory automorphism of $\mathrm{GL}(n, \mathbb{C})$. Define

$$\sigma_J(g) = J^{-1}(g^t)^{-1} J \qquad \text{for } g \in \mathrm{GL}(n, \mathbb{C}).$$

Then the group $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ is the set of *fixed points* of σ_J:

$$g \in \mathrm{Sp}(\mathbb{C}^{2l}, \Omega) \Longleftrightarrow \sigma_J(g) = g.$$

We say that a basis $\{v_1, \ldots, v_l, v_{-1}, \ldots, v_{-l}\}$ for $\mathbb{C}^{2l}$ is an Ω-*symplectic basis* if $\Omega(v_i, v_j) = 0$ for $i \neq -j$ and $\Omega(v_i, v_{-i}) = 1$ for $i = 1, \ldots, l$. When

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix},$$

where I is the $l \times l$ identity matrix, then $v_1 = e_1, \ldots, v_l = e_l, v_{-1} = e_{l+1}, \ldots, v_{-l} = e_{2l}$ is an Ω-symplectic basis for $\mathbb{C}^{2l}$. Hence $g \in \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ in this case precisely when the vectors $w_i = g v_i$ make up an Ω-symplectic basis for $\mathbb{C}^n$.

Lemma 1.1.4 *Let Ω be a nondegenerate skew form on $\mathbb{C}^{2l}$. Suppose*

$$\{v_1, \ldots, v_l, v_{-1}, \ldots, v_{-l}\} \qquad \text{and} \qquad \{v'_1, \ldots, v'_l, v'_{-1}, \ldots, v'_{-l}\}$$

are Ω-symplectic bases for $\mathbb{C}^{2l}$. Let $g \in \mathrm{GL}(2l, \mathbb{C})$ be defined by $g v_i = v'_i$ for $i = \pm 1, \ldots, \pm l$. Then $g \in \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$.

Proof. The proof is same as for Lemma 1.1.1. ■

Proposition 1.1.5 *Let Ω and Ω' be nondegenerate skew-symmetric bilinear forms on $\mathbb{C}^{2l}$. Then there exists $\gamma \in \mathrm{GL}(2l, \mathbb{C})$ such that $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega') = \gamma \mathrm{Sp}(\mathbb{C}^{2l}, \Omega) \gamma^{-1}$.*

Proof. Use Lemma B.2.4 to construct canonical symplectic bases relative to Ω and Ω' and let γ be the transformation from the Ω basis to the Ω' basis. Then argue as in the proof of Proposition 1.1.2. ■

We call $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ a group of *type* C_l. When Ω is a nondegenerate skew-symmetric bilinear form on a finite dimensional complex vector space V, we shall write $\mathrm{Sp}(V, \Omega)$ for the corresponding symplectic group. If the choice of Ω is understood from the context or is irrelevant, we shall denote this group as $\mathrm{Sp}(V)$. When $V = \mathbb{C}^{2l}$ we will generally write $\mathrm{Sp}(l, \mathbb{C})$ or $\mathrm{Sp}_{2l}(\mathbb{C})$ for any element of the corresponding isomorphism class of linear algebraic groups.

The groups $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$, and $\mathrm{Sp}(l, \mathbb{C})$ are called the *classical groups*.

1.1.2 Regular Functions

We now establish some basic properties of linear algebraic groups. We begin with the notion of *regular function*. For the group $\mathrm{GL}(n, \mathbb{C})$, the ring of *regular functions* is defined as

$$\mathrm{Aff}(\mathrm{GL}(n, \mathbb{C})) = \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}, (\det)^{-1}].$$

This is the $\mathbb{C}$ algebra generated by the matrix entry functions x_{ij} and the function $(\det)^{-1}$. In the language of algebraic geometry, $\mathrm{GL}(n, \mathbb{C})$ is the principal open set $\{g \in M_n(\mathbb{C}) : \det(g) \neq 0\}$ in the vector space $M_n(\mathbb{C})$, and $\mathrm{Aff}(G)$ is the ring of regular functions on this principal open set (see Sections A.1.1 and A.1.4).

For any complex vector space V, $\mathrm{End}(V)$ denotes the algebra of all linear transformations on V and $\mathrm{GL}(V)$ denotes the group of invertible linear transformations on V. Suppose $\dim V = n$ and $\{e_i\}$ is a basis for V. For $g \in \mathrm{GL}(V)$ let $\phi(g)$ be the matrix $[g_{ij}]$, where

$$ge_j = \sum_{i=1}^{n} g_{ij} e_i.$$

The map $g \mapsto \phi(g)$ gives an isomorphism $\phi : \mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{C})$. We define the *regular functions* on $\mathrm{GL}(V)$ to be those of the form $f \circ \phi$, where f is a regular function on $\mathrm{GL}(n, \mathbb{C})$. We denote by $\mathrm{Aff}(\mathrm{GL}(V))$ the algebra of regular functions on $\mathrm{GL}(V)$. This does not depend on the choice of basis for V.

Given $B \in \mathrm{End}(V)$, we define a function f_B on $\mathrm{End}(V)$ by

$$f_B(Y) = \mathrm{tr}_V(YB), \qquad \text{for } Y \in \mathrm{End}(V). \tag{1.1.4}$$

Taking $V = \mathbb{C}^n$ and $B = E_{ij}$, the elementary matrix with 1 in the i, j entry and 0 elsewhere, we have $f_{E_{ij}}(Y) = x_{ji}(Y)$. Since the map $B \to f_B$ is linear, we see that the function f_B on $\mathrm{GL}(n, \mathbb{C})$ is a linear combination of the matrix-entry functions and hence is regular. Furthermore, the algebra $\mathrm{Aff}(\mathrm{GL}(n, \mathbb{C}))$ is generated by $\{f_B : B \in M_n(\mathbb{C})\}$ and $(\det)^{-1}$. Thus for any finite-dimensional vector space V the algebra $\mathrm{Aff}(\mathrm{GL}(V))$ is generated by $(\det)^{-1}$ and the functions f_B for $B \in \mathrm{End}(V)$.

The definition of a linear algebraic group can be rephrased in a basis-free way as follows: Let V be a finite-dimensional complex vector space. A subgroup $G \subset \mathrm{GL}(V)$

is a linear algebraic group if G is a closed subset of $\mathrm{GL}(V)$, relative to the Zariski topology (see Section A.1.2). To see that this agrees with the definition in Section 1.1.1, we observe that the Zariski-closed subsets of $\mathrm{GL}(V)$ are defined by equations of the form

$$f(x_{11}(g), x_{12}(g), \ldots, x_{nn}(g), \det(g)^{-1}) = 0,$$

where f is a polynomial in $n^2 + 1$ variables. Since $\det(g) \neq 0$, we can multiply this equation by $\det(g)^k$ for a suitably large k to obtain a polynomial equation in the matrix coefficients of g.

A complex-valued function f on G is called *regular* if it is the restriction to G of a regular function on $\mathrm{GL}(V)$. The set $\mathrm{Aff}(G)$ of regular functions on G is a commutative algebra over $\mathbb{C}$ under pointwise multiplication. It has a finite set of generators, namely the restrictions to G of $(\det)^{-1}$ and the functions f_B, with B varying over any linear basis for $\mathrm{End}(V)$. Define

$$\mathcal{I}_G = \{f \in \mathrm{Aff}(\mathrm{GL}(V)) : f(G) = 0\}.$$

This is an ideal in $\mathrm{Aff}(\mathrm{GL}(V))$ that we can describe in terms of polynomials on $\mathrm{End}(V)$ by

$$\mathcal{I}_G = \bigcup_{p \geq 0} \{(\det)^{-p} f : f \in \mathrm{Aff}(\mathrm{End}(V)),\ f(G) = 0\}. \tag{1.1.5}$$

The map $f \mapsto f|_G$ gives a ring isomorphism

$$\mathrm{Aff}(G) \cong \mathrm{Aff}(\mathrm{GL}(V))/\mathcal{I}_G. \tag{1.1.6}$$

If G, H are linear algebraic groups, then an (abstract) group homomorphism $\phi : G \to H$ is *regular* if $\phi^*(\mathrm{Aff}(H)) \subset \mathrm{Aff}(G)$. Here for $f \in \mathrm{Aff}(H)$ we set $\phi^*(f)(g) = f(\phi(g))$. We say that G and H are *isomorphic as algebraic groups* if there exists a regular homomorphism $\phi : G \to H$ that has a regular inverse.

Examples

1. Let D_n be the subgroup of diagonal matrices in $\mathrm{GL}(n, \mathbb{C})$. The functions $x_{ij} \in \mathcal{I}_{D_n}$ for $i \neq j$, and $(\det)^{-1}|_{D_n} = (x_{11} \cdots x_{nn})^{-1}$. The functions $x_i = x_{ii}|_{D_n}$ are algebraically independent, and we have

$$\mathrm{Aff}(D_n) = \mathbb{C}[x_1, \ldots, x_n, (x_1 \cdots x_n)^{-1}] = \mathbb{C}\big[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\big],$$

which is the ring of finite Laurent polynomials in n variables. We call an algebraic group H that is isomorphic to D_n an *algebraic torus* of *rank* n. Since the algebra $\mathrm{Aff}(H)$ is isomorphic to $\mathrm{Aff}(D_n)$ in this case, it is clear that the rank is well defined.

2. Let N_n be the subgroup of upper-triangular matrices with unit diagonal in $\mathrm{GL}(n, \mathbb{C})$. The functions $x_{ij} \in \mathcal{I}_{N_n}$ for $i > j$, and $x_{ii} = 1$, $\det = 1$ on N_n. Thus

$$\mathrm{Aff}(N_n) = \mathbb{C}[x_{12}, x_{13}, \ldots, x_{n-1,n}]$$

is the ring of polynomials in the variables $\{x_{ij} : i < j\}$.

Let G be a linear algebraic group. The set $G \times G$ carries the structure of an affine algebraic set, with the algebra of regular functions

$$\mathrm{Aff}(G \times G) \cong \mathrm{Aff}(G) \otimes \mathrm{Aff}(G).$$

In this isomorphism, $f' \otimes f'' \in \mathrm{Aff}(G) \otimes \mathrm{Aff}(G)$ is identified with the function $(g, h) \mapsto f'(g) f''(h)$ on $G \times G$ (see Lemma A.1.10).

Proposition 1.1.6 *The maps $\mu : G \times G \to G$ and $\iota : G \to G$ given by multiplication and inversion are regular. If $f \in \mathrm{Aff}(G)$ then there exists an integer p and $f_i', f_i'' \in \mathrm{Aff}(G)$ for $i = 1, \ldots, p$, such that*

$$f(gh) = \sum_{i=1}^{p} f_i'(g)\, f_i''(h) \qquad \text{for } g, h \in G. \tag{1.1.7}$$

Furthermore, for fixed $g \in G$ the maps $x \mapsto L_g(x) = gx$ and $x \mapsto R_g(x) = xg$ from $G \to G$ are regular.

Proof. Fix a basis for V, and let $\{x_{ij}\}$ be the matrix entry functions on $\mathrm{End}(V)$ corresponding to this basis. Cramer's rule says that $g^{-1} = \det(g)^{-1} \mathcal{A}(g)$, where $\mathcal{A}(g)$ is the classical adjoint matrix (transposed cofactor matrix) of g. Since the matrix entries of $\mathcal{A}(g)$ are polynomials in the x_{ij}, it is clear from (1.1.6) that $\iota^* f \in \mathrm{Aff}(G)$ whenever $f \in \mathrm{Aff}(G)$.

Let $g, h \in G$. Then

$$x_{ij}(gh) = \sum_r x_{ir}(g)\, x_{rj}(h).$$

Hence (1.1.7) is valid when $f = x_{ij}|_G$. It also holds when $f = (\det)^{-1}|_G$ by the multiplicative property of the determinant. Let $\mathcal{A}$ be the set of $f \in \mathrm{Aff}(G)$ for which (1.1.7) is valid. Then $\mathcal{A}$ is a subalgebra of $\mathrm{Aff}(G)$, and we have just verified that the matrix entry functions and $\det^{-1}$ are in $\mathcal{A}$. Since these functions generate $\mathrm{Aff}(G)$ as an algebra, it follows that $\mathcal{A} = \mathrm{Aff}(G)$.

Using the identification $\mathrm{Aff}(G \times G) = \mathrm{Aff}(G) \otimes \mathrm{Aff}(G)$, we can write (1.1.7) as

$$\mu^*(f) = \sum_i f_i' \otimes f_i''.$$

This shows that μ is a regular map. Similarly,

$$L_g^*(f) = \sum_i f_i'(g) f_i'', \qquad R_g^*(f) = \sum_i f_i''(g) f_i',$$

which proves that L_g and R_g are regular. ∎

If $G \subset \mathrm{GL}(V)$, $H \subset \mathrm{GL}(W)$ are linear algebraic groups, then we make the group-theoretic direct product $K = G \times H$ into an algebraic group by the natural block diagonal embedding into $\mathrm{GL}(V \oplus W)$ as the elements

$$k = \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} \qquad g \in G,\ h \in H$$

(such elements comprise a closed subgroup of $\mathrm{GL}(V \oplus W)$). This embedding defines a map

$$\mathrm{Aff}(K) \to \mathrm{Aff}(G) \otimes \mathrm{Aff}(H),$$

which is easily seen to be an isomorphism. Hence K is also the product of G and H as an affine algebraic set (cf. Appendix A, Lemma A.1.10).

1.1.3 Representations

Let G be a linear algebraic group. A *representation* of G is a pair (ρ, V), where V is a complex vector space (not necessarily finite-dimensional), and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. We say that the representation is *regular* if $\dim V < \infty$ and the functions on G

$$g \mapsto \langle \rho(g)v, v^* \rangle, \tag{1.1.8}$$

which we call *matrix coefficients* of ρ, are regular, for all $v \in V$ and $v^* \in V^*$. (In other words, ρ is a regular homomorphism from G to $\mathrm{GL}(V)$.) If we fix a basis for V and write out the matrix for $\rho(g)$ in this basis,

$$\rho(g) = \begin{bmatrix} \rho_{11}(g) & \cdots & \rho_{1d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d1}(g) & \cdots & \rho_{dd}(g) \end{bmatrix},$$

then all the functions $\rho_{ij}(g)$ on G are regular.

It will be convenient to phrase the definition of regularity as follows: On $\mathrm{End}(V)$ we have the symmetric bilinear form $A, B \mapsto \mathrm{tr}_V(AB)$. This form is nondegenerate, so if $\lambda \in \mathrm{End}(V)^*$ then there exists $A_\lambda \in \mathrm{End}(V)$ such that $\lambda(X) = \mathrm{tr}_V(A_\lambda X)$. For $B \in \mathrm{End}(V)$ define the function f_B^ρ on G by

$$f_B^\rho(g) = \mathrm{tr}_V(\rho(g)B)$$

(when B has rank one, then this function is of the form (1.1.8)). Then (ρ, V) is regular if and only if f_B^ρ is a regular function on G, for all $B \in \mathrm{End}(V)$. We set

$$E^\rho = \left\{ f_B^\rho : B \in \mathrm{End}(V) \right\}.$$

This is the linear space spanned by the functions in the matrix for ρ. It is invariant under left and right translations by G, and we call it the space of *representative functions* associated with ρ.

When V is infinite-dimensional, we say that a representation (ρ, V) is *locally regular* if for any finite-dimensional subspace $F \subset V$ there exists a finite-dimensional subspace $E \supset F$ such that $\rho(G)E \subset E$ and the restriction of ρ to E is a regular representation.

If (ρ, V) is a regular representation and $W \subset V$ is a linear subspace, then we say that W is G*-invariant* if $\rho(g)w \in W$ for all $g \in G$ and $w \in W$. In this case we obtain a representation σ of G on W by restriction of $\rho(g)$. We also obtain a representation

τ of G on the quotient space V/W by setting $\tau(g)(v+W) = \rho(g)v + W$. If we take a basis for W and complete it to a basis for V, then the matrix of $\rho(g)$ relative to this basis has the block form

$$\rho(g) = \begin{bmatrix} \sigma(g) & * \\ 0 & \tau(g) \end{bmatrix} \tag{1.1.9}$$

(with the basis for W listed first). This matrix form shows that the representations (σ, W) and $(\tau, V/W)$ are regular.

If (ρ, V) and (τ, W) are representations of G, then we say that they are *equivalent* if there is a linear bijection $T : V \to W$ so that

$$T\rho(g)T^{-1} = \tau(g) \qquad \text{for all } g \in G.$$

In this case we write $\rho \cong \tau$.

We say that a representation (ρ, V) with $V \neq \{0\}$ is *reducible* if there is a G-invariant subspace $W \subset V$ such that $W \neq \{0\}$ and $W \neq V$. This means that there exists a basis for V so that $\rho(g)$ has the block form (1.1.9) with all blocks of size at least 1×1. A representation that is not reducible is called *irreducible*.

Examples

1. Let $G \subset \mathrm{GL}(V)$ be a linear algebraic group. By definition of $\mathrm{Aff}(G)$, the representation $\rho(g) = g$ on V is regular. We call ρ the *defining* representation of G.
2. Let (ρ, V) be a regular representation. Define the *contragredient* (or *dual*) representation (ρ^*, V^*) by $\rho^*(g)v^* = v^* \circ \rho(g^{-1})$. Then ρ^* is obviously regular since

$$\langle \rho(g)v, v^* \rangle = \langle v, \rho^*(g^{-1})v^* \rangle$$

for $v \in V$ and $v^* \in V^*$. The space of representative functions for ρ^* consists of the functions $g \mapsto f(g^{-1})$, where f is a representative function for ρ. If $W \subset V$ is a G-invariant subspace, then

$$W^{\perp} = \{v^* \in V^* : \langle v^*, w \rangle = 0 \text{ for all } w \in W\}$$

is a G-invariant subspace of V^*. In particular, if (ρ, V) is irreducible then so is (ρ^*, V^*). The natural vector-space isomorphism $(V^*)^* \cong V$ gives an equivalence $(\rho^*)^* \cong \rho$.
3. Let (ρ, V) and (σ, W) be regular representations of G. Define the *direct sum* representation $\rho \oplus \sigma$ on $V \oplus W$ by

$$(\rho \oplus \sigma)(g)(v \oplus w) = \rho(g)v \oplus \sigma(g)w$$

for $g \in G$, $v \in V$, and $w \in W$. Then $\rho \oplus \tau$ is obviously a representation of G. It is regular since

$$\langle (\rho \oplus \sigma)(g)(v \oplus w), v^* \oplus w^* \rangle = \langle \rho(g)v, v^* \rangle + \langle \sigma(g)w, w^* \rangle$$