1 Introduction

The theory of elliptic partial differential equations has its origins in the eighteenth century, and the present chapter outlines a few of the most important historical developments up to the beginning of the twentieth century. We concentrate on those topics that will play an important role in the main part of the book, and change the notation of the original authors, wherever necessary, to achieve consistency with what comes later. Such a brief account cannot pretend to be a balanced historical survey, but this chapter should at least serve to introduce the main ideas of the book in a readable manner.

To limit subsequent interruptions, we fix some notational conventions at the outset. Let Ω denote a bounded, open subset of $\mathbb{R}^n$ (where $n = 2$ or 3 in this chapter), and assume that the boundary $\Gamma = \partial\Omega$ is sufficiently regular for the outward unit normal ν and the element of surface area $d\sigma$ to make sense. Given a function u defined on Ω, we denote the normal derivative by $\partial_\nu u$ or $\partial u/\partial \nu$. Sometimes we shall work with both the interior and the exterior domains (see Figure 1)

$$\Omega^- = \Omega \quad \text{and} \quad \Omega^+ = \mathbb{R}^n \backslash (\Omega^- \cup \Gamma),$$

in which case, if the function u is defined on $\Omega^\pm$, we write

$$\gamma^\pm u(x) = \lim_{y \to x,\, y \in \Omega^\pm} u(y) \quad \text{and}$$
$$\partial_\nu^\pm u(x) = \lim_{y \to x,\, y \in \Omega^\pm} \nu(x) \cdot \operatorname{grad} u(y) \qquad \text{for } x \in \Gamma,$$

whenever these limits exist. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted by $|x|$.

The prototype of an elliptic partial differential equation is $\Delta u = 0$, where Δ denotes the Laplace operator (or Laplacian), defined, in n dimensions, by

$$\Delta u(x) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}. \tag{1.1}$$

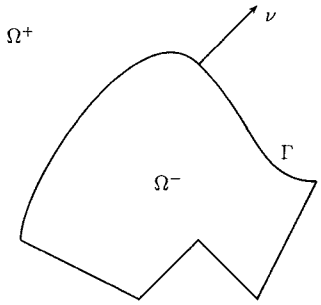

Figure 1. Interior and exterior domains Ω^- and Ω^+ with boundary Γ.

When $\Delta u = 0$ on Ω, we say that the function u is *harmonic* on Ω. In two dimensions, there is a close connection between the Laplace equation and complex-analytic functions. Indeed, $u + \mathrm{i}v$ is differentiable as a function of the complex variable $x_1 + \mathrm{i}x_2$ if and only if u and v satisfy the Cauchy–Riemann equations,

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2} \quad \text{and} \quad \frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}, \tag{1.2}$$

in which case $\Delta u = 0 = \Delta v$ and we say that u and v are *conjugate harmonic functions*.

The pair of equations (1.2) appeared in Jean-le-Rond d'Alembert's *Essai d'une Nouvelle Théorie de la Résistance des Fluides*, published in 1752. At around the same time, Leonhard Euler derived the equations of motion for an irrotational fluid in three dimensions. He showed that the fluid velocity has the form $\operatorname{grad} u$, and that for a steady flow the velocity potential satisfies $\Delta u = 0$. This work of d'Alembert and Euler is discussed by Truesdell [100]; see also Dauben [18, p. 311].

In 1774, Joseph-Louis Lagrange won the *Prix de l'Academie Royale des Sciences* for a paper [51] on the motion of the moon; see also [30, pp. 478–479, 1049]. This paper drew attention to two functions that later came to be known as the *fundamental solution*,

$$G(x, y) = \frac{1}{4\pi |x - y|} \qquad \text{for } x, y \in \mathbb{R}^3 \text{ and } x \neq y, \tag{1.3}$$

and the *Newtonian potential*,

$$u(x) = \int_{\mathbb{R}^3} G(x, y) f(y)\, dy. \tag{1.4}$$

Up to an appropriate constant of proportionality, $G(x, y)$ is the gravitational

potential at x due to a unit point mass at y, and thus u is the gravitational potential due to a continuous mass distribution with density f. The Coulomb force law in electrostatics has the same inverse-square form as Newton's law of gravitational attraction. Thus, u also describes the electrostatic potential due to a charge distribution with density f; mathematically, the only change is that f may be negative.

In a paper of 1782 entitled *Théorie des attractions des sphéroides et de la figure des planètes*, Pierre Simon de Laplace observed that the Newtonian potential (1.4) satisfies $\Delta u = 0$ outside the support of f, writing Δu in spherical polar coordinates. Later, in a paper of 1787 on the rings of Saturn, he gave the same result in Cartesian and cylindrical coordinates. Birkhoff and Merzbach [7, pp. 335–338] give English translations of relevant excerpts from these two works.

By transforming to polar coordinates centred at x, i.e., by using the substitution $y = x + \rho\omega$ where $\rho = |y - x|$, it is easy to see that the Newtonian potential (1.4) makes sense even if x lies within the support of f, because $dy = \rho^2\, d\rho\, d\omega$. However, the second partial derivatives of G are $O(\rho^{-3})$, and this singularity is too strong to allow a direct calculation of Δu by simply differentiating under the integral sign. In fact, it turns out that

$$-\Delta u = f$$

everywhere on $\mathbb{R}^3$, an equation derived by Siméon-Denis Poisson [7, pp. 342–346] in 1813; see Exercise 1.1 for the special case when f is radially symmetric.

Poisson made other important contributions to potential theory. A paper [18, p. 360] of 1812 dealt with the distribution of electric charge on a conductor Ω. In equilibrium, mutual repulsion causes all of the charge to reside on the surface Γ of the conducting body, and Γ is an equipotential surface. The electrical potential at $x \in \mathbb{R}^3$ due to a charge distribution with surface density ψ on Γ is given by the integral

$$\mathrm{SL}\,\psi(x) = \int_\Gamma G(x, y)\psi(y)\, d\sigma_y, \tag{1.5}$$

so $\mathrm{SL}\,\psi$ is constant on Γ if ψ is the equilibrium distribution. The function $\mathrm{SL}\,\psi$ is known as the *single-layer potential* with density ψ, and satisfies the Laplace equation on the complement of Γ, i.e., on $\Omega^+ \cup \Omega^-$. Although $\mathrm{SL}\,\psi$ is continuous everywhere, Poisson found that its normal derivative has a jump discontinuity:

$$\partial_\nu^+ \mathrm{SL}\,\psi - \partial_\nu^- \mathrm{SL}\,\psi = -\psi \quad \text{on } \Gamma. \tag{1.6}$$

Exercise 1.2 proves an easy special case of this result.

A further stimulus to the study of the Laplace equation was Jean-Baptiste-Joseph Fourier's theory of heat diffusion. In 1807, he published a short note containing the heat equation,

$$\frac{\partial u}{\partial t} - a\triangle u = 0,$$

where $u = u(x,t)$ is the temperature at position x and time t, and $a > 0$ is the thermal conductivity (here assumed constant). For a body Ω in thermal equilibrium, $\partial u/\partial t = 0$, so if one knows the temperature distribution g on the bounding surface Γ, then one can determine the temperature distribution u in the interior by solving the boundary value problem

$$\begin{aligned} \triangle u &= 0 \quad \text{on } \Omega, \\ u &= g \quad \text{on } \Gamma. \end{aligned} \tag{1.7}$$

This problem later became known as the *Dirichlet problem*, and for particular, simple choices of Ω, Fourier constructed solutions using separation of variables; see [7, pp. 132–138]. His book, *Théorie analytique de la chaleur*, was published in 1822.

In 1828, George Green published *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* [31], [33, pp. 1–115]; an extract appears in [7, pp. 347–358]. In his introduction, Green discusses previous work by other authors including Poisson, and writes that

> although many of the artifices employed in the works before mentioned are remarkable for their elegance, it is easy to see they are adapted only to particular objects, and that some general method, capable of being employed in every case, is still wanting.

Green's "general method" was based on his two integral identities:

$$\int_\Omega \operatorname{grad} w \cdot \operatorname{grad} u \, dx = \int_\Gamma w \frac{\partial u}{\partial \nu} \, d\sigma - \int_\Omega w \triangle u \, dx \tag{1.8}$$

and

$$\int_\Omega (w\triangle u - u\triangle w)\, dx = \int_\Gamma \left(w\frac{\partial u}{\partial \nu} - u\frac{\partial w}{\partial \nu} \right) d\sigma, \tag{1.9}$$

where u and w are arbitrary, sufficiently regular functions. Using (1.9) with

$w(y) = G(x, y)$, he obtained a third identity,

$$u(x) = -\int_\Omega G(x, y)\Delta u(y)\, dy - \int_\Gamma u(y)\frac{\partial}{\partial \nu_y} G(x, y)\, d\sigma_y + \int_\Gamma G(x, y)\frac{\partial u}{\partial \nu}(y)\, d\sigma_y \quad \text{for } x \in \Omega. \tag{1.10}$$

Actually, Green derived a more general result, showing that (1.10) is valid when $G(x, y)$ is replaced by a function of the form

$$\mathrm{Gr}(x, y) = G(x, y) + V(x, y),$$

where V is any smooth function satisfying $\Delta_y V(x, y) = 0$ for $x, y \in \Omega$. In other words, $\mathrm{Gr}(x, y)$ has the same singular behaviour as $G(x, y)$ when $y = x$, and satisfies $\Delta_y \mathrm{Gr}(x, y) = 0$ for $y \neq x$. Green gave a heuristic argument for the existence of a unique such Gr satisfying $\mathrm{Gr}(x, y) = 0$ for all $y \in \Gamma$: physically, $\mathrm{Gr}(x, y)$ represents the electrostatic potential at y due to a point charge at x when Γ is an earthed conductor. This particular Gr became known as the *Green's function* for the domain Ω, and yields an integral representation formula for the solution of the Dirichlet problem (1.7),

$$u(x) = -\int_\Gamma g(y)\frac{\partial}{\partial \nu_y}\mathrm{Gr}(x, y)\, d\sigma_y \quad \text{for } x \in \Omega. \tag{1.11}$$

In practice, finding an explicit formula for Gr is possible only for very simple domains. For instance, if Ω is the open ball with radius $r > 0$ centred at the origin, then

$$\mathrm{Gr}(x, y) = \frac{1}{4\pi|x - y|} - \frac{1}{4\pi}\frac{r}{|x||x^\sharp - y|},$$

where $x^\sharp = (r/|x|)^2 x$ is the image of x under a reflection in the sphere Γ. In this case, the integral (1.11) is given by

$$u(x) = \frac{1}{4\pi r}\int_{|y|=r} g(y)\frac{r^2 - |x|^2}{|x - y|^3}\, d\sigma_y \quad \text{for } |x| < r,$$

a formula obtained by Poisson [18, p. 360] in 1813 by a different method. Green also used (1.10) to derive a kind of converse to the jump relation (1.6), by showing that if a function u satifies the Laplace equation on $\Omega^+ \cup \Omega^-$, is continuous everywhere and decays appropriately at infinity, then $u = \mathrm{SL}\,\psi$, where $\psi = -(\partial_\nu^+ u - \partial_\nu^- u)$.

Green's *Essay* did not begin to become widely known until 1845, when William Thomson (Lord Kelvin) introduced it to Joseph Liouville in Paris [99, pp. 113–121]. Eventually, Thomson had the work published in three parts during 1850–1854 in Crelle's *Journal für die reine und angewandte Mathematik* [31].

Meanwhile, Poisson and others continued to apply the method of separation of variables to a variety of physical problems. A key step in many such calculations is to solve a two-point boundary value problem with a parameter $\lambda > 0$,

$$\begin{aligned} -\frac{d}{dx}\left(a\frac{du}{dx}\right) + (b - \lambda w)u &= 0 \quad \text{for } 0 < x < 1, \\ \frac{du}{dx} - m_0 u &= 0 \quad \text{at } x = 0, \\ \frac{du}{dx} - m_1 u &= 0 \quad \text{at } x = 1, \end{aligned} \tag{1.12}$$

where a, b and w are known real-valued functions of x such that $a > 0$ and $w > 0$, and where m_0 and m_1 are known constants (possibly ∞, in which case the boundary condition is to be interpreted as $u = 0$). The main features of the problem (1.12) can be seen in the simplest example: $a = w = 1$ and $b = 0$. The general solution of the differential equation is then a linear combination of $\sin(\sqrt{\lambda}x)$ and $\cos(\sqrt{\lambda}x)$, and the boundary conditions imply that the solution is identically zero unless the parameter λ satisfies a certain transcendental equation having a sequence of positive solutions $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ with $\lambda_j \to \infty$. In the general case, the number λ_j was subsequently called an *eigenvalue* for the problem, and any corresponding, non-trivial solution $u = \phi_j$ of the differential equation was called an *eigenfunction*. For the special case $a = w = 1$, Poisson showed in 1826 that eigenfunctions corresponding to distinct eigenvalues are orthogonal, i.e.,

$$\int_0^1 \phi_j(x)\phi_k(x)w(x)\,dx = 0 \quad \text{if } \lambda_j \ne \lambda_k,$$

and that all eigenvalues are real; see [62, p. 433]. A much deeper analysis was given by Charles François Sturm in 1836, who established many important properties of the eigenfunctions, as well as proving the existence of infinitely many eigenvalues. Building on Sturm's work, Liouville showed in two papers from 1836 and 1837 that an arbitrary function f could be expanded in a generalised

Fourier series,

$$f(x) = \sum_{j=1}^{\infty} c_j \phi_j(x), \qquad \text{where } c_j = \frac{\int_0^1 \phi_j(x) f(x) w(x)\, dx}{\int_0^1 \phi_j(x)^2 w(x)\, dx},$$

thereby justifying many applications of the method of separation of variables. Excerpts from the papers of Sturm and Liouville are reproduced in [7, pp. 258–281]; see also [62, Chapter X].

Carl Friedrich Gauss wrote a long paper [26] on potential theory in 1839; see also [7, pp. 358–361] and Lützen [62, pp. 583–586]. He re-derived many of Poisson's results, including (1.6), using more rigorous arguments, and was apparently unaware of Green's work. Gauss sought to find, for an arbitrary conductor Ω, the equilibrium charge distribution with total charge M, i.e., in mathematical terms, he sought to find a function ψ whose single-layer potential $\mathrm{SL}\,\psi$ is constant on Γ, subject to the constraint that $\int_\Gamma \psi\, d\sigma = M$. Introducing an arbitrary function g, he considered the quadratic functional

$$J_g(\phi) = \int_\Gamma (S\phi - 2g)\phi\, d\sigma,$$

where $S\phi = \gamma^+ \mathrm{SL}\,\phi = \gamma^- \mathrm{SL}\,\phi$ denotes the boundary values of the single-layer potential, or, explicitly,

$$S\phi(x) = \int_\Gamma G(x, y)\phi(y)\, d\sigma_y \quad \text{for } x \in \Gamma.$$

In the case $g = 0$, the quantity $J_g(\phi)$ has a physical meaning: it is proportional to the self-energy of the charge distribution ϕ; see Kellogg [45, pp. 79–80].

One easily sees that $J_g(\phi)$ is bounded below for all ϕ in the class V_M of functions satisfying $\int_\Gamma \phi\, d\sigma = M$ and $\phi \geq 0$ on Γ. Also, $J_g(\phi + \delta\phi) = J_g(\phi) + \delta J_g + O(\delta\phi^2)$, where the first variation of J_g is given by

$$\delta J_g = \delta J_g(\phi, \delta\phi) = 2\int_\Gamma (S\phi - g)\, \delta\phi\, d\sigma.$$

Suppose that the minimum value of J_g over the class V_M is achieved when $\phi = \psi$. It follows that $\delta J_g(\psi, \delta\phi) = 0$ for all $\delta\phi$ satisfying $\int_\Gamma \delta\phi\, d\sigma = 0$ and $\psi + \delta\phi \geq 0$ on Γ, and therefore $S\psi - g$ is constant on any component of Γ where $\psi > 0$. Gauss showed that if $g = 0$, then $\psi > 0$ everywhere on Γ, and thus deduced the existence of an equilibrium potential from the existence of a minimiser for J_0. He also showed that this minimiser is unique,

and gave an argument for the existence of a solution ψ to the boundary integral equation

$$S\psi = g \quad \text{on } \Gamma. \tag{1.13}$$

The single-layer potential of this ψ is the solution of the Dirichlet problem for the Laplace equation, i.e., $u = \mathrm{SL}\,\psi$ satisfies (1.7).

In a series of papers from 1845 to 1846, Liouville studied the single-layer potential when Γ is an ellipsoid, solving the integral equation (1.13) by adapting his earlier work on the eigenvalue problem (1.12). Let w be the equilibrium density for Γ, normalised so that $Sw = 1$. Liouville showed that if Γ is an ellipsoid, then

$$S(w\psi_j) = \mu_j \psi_j \quad \text{for } j = 1, 2, 3, \ldots,$$

where the ψ_j are *Lamé functions*, and the μ_j are certain constants satisfying

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots > 0 \qquad \text{with } \mu_j \to 0 \text{ as } j \to \infty.$$

He established the orthogonality property

$$\int_\Gamma \psi_j(x)\psi_k(x)w(x)\,d\sigma_x = 0 \quad \text{for } j \neq k,$$

and concluded that the solution of (1.13) is

$$\psi(x) = w(x)\sum_{j=1}^{\infty} c_j \psi_j(x), \qquad \text{where } c_j = \frac{\int_\Gamma \psi_j(x)g(x)w(x)\,d\sigma_x}{\int_\Gamma \psi_j(x)^2 w(x)\,d\sigma_x}.$$

In his unpublished notebooks (described in [62, Chapter XV]) Liouville went a considerable distance towards generalising these results to the case of an arbitrary surface Γ, inventing in the process the Rayleigh–Ritz procedure for finding the eigenvalues and eigenfunctions, 20 years before Rayleigh [97] and 60 years before Ritz [98].

During the 1840s, Thomson and Peter Gustav Lejeune Dirichlet separately advanced another type of existence argument [7, pp. 379–387] that became widely known on account of its use by Riemann in his theory of complex analytic functions. Riemann introduced the term *Dirichlet's principle* for this method of establishing the existence of a solution to the Dirichlet problem, although a related variational argument had earlier been used by Green [32]. If

one considers the functional

$$J(v) = \int_\Omega |\operatorname{grad} v|^2 \, dx$$

for v in a class of sufficiently regular functions V_g satisfying $v = g$ on Γ, then it seems obvious, because $J(v) \geq 0$ for all $v \in V_g$, that there exists a $u \in V_g$ satisfying

$$J(u) \leq J(v) \quad \text{for all } v \in V_g. \tag{1.14}$$

Given any w such that $w = 0$ on Γ, and any constant h, the function $v = u + hw$ belongs to V_g, and, assuming the validity of the first Green identity (1.8), simple manipulations yield

$$J(v) = J(u) - 2h \int_\Omega w \Delta u \, dx + h^2 J(w).$$

Here, the constant h is arbitrary, so the minimum condition (1.14) implies that

$$\int_\Omega w \Delta u \, dx = 0 \quad \text{whenever } w = 0 \text{ on } \Gamma.$$

By choosing w to take the same sign as Δu throughout Ω, we conclude that u is a solution of the Dirichlet problem for the Laplace equation. Conversely, each solution of the Dirichlet problem minimises the integral. Dirichlet also established the uniqueness of the minimiser u. In fact, if both u_1 and u_2 minimise J in the class of functions V_g, then the difference $w = u_1 - u_2$ vanishes on Γ, and, arguing as above with $h = 1$, we find that $J(u_1) = J(u_2) + J(w)$. Thus, $J(w) = 0$, so w is constant, and hence identically zero, implying that $u_1 = u_2$ on Ω.

In 1869, H. Weber [104] employed the quadratic functional $J(v)$ in a Rayleigh–Ritz procedure to show the existence of eigenfunctions and eigenvalues for the Laplacian on a general bounded domain. He minimised $J(v)$ subject to *two* constraints: $v = 0$ on Γ, and $\int_\Omega v(x)^2 \, dx = 1$. If we suppose that a minimum is achieved when $v = u_1$, then by arguing as above we see that

$$\int_\Omega w \Delta u_1 \, dx = 0 \qquad \text{whenever } w = 0 \text{ on } \Gamma \quad \text{and} \quad \int_\Omega w(x) u_1(x) \, dx = 0.$$

Here, the extra restriction on w arises from the second of the constraints in the minimisation problem. Weber showed that $-\Delta u_1 = \lambda_1 u_1$ on Ω, where $\lambda_1 = J(u_1)$. In fact, for an arbitrary v satisfying $v = 0$ on Γ, if we put

$a = \int_\Omega v u_1\,dx$ and $w = v - a u_1$, then $w = 0$ on Γ, and $\int_\Omega w u_1\,dx = 0$, so by the first Green identity,

$$\int_\Omega v(-\triangle u_1)\,dx = -\int_\Omega (w + a u_1)\triangle u_1\,dx$$
$$= a\left(J(u_1) - \int_\Gamma u_1 \frac{\partial u_1}{\partial \nu}\,d\sigma\right) = \lambda_1 \int_\Omega v u_1\,dx,$$

remembering that $u_1 = 0$ on Γ. Next, Weber minimised $J(v)$ subject to three constraints: the two previous ones and in addition $\int_\Omega v u_1\,dx = 0$. The minimiser u_2 is the next eigenfunction, satisfying $-\triangle u_2 = \lambda_2 u_2$ on Ω, where $\lambda_2 = J(u_2) \geq \lambda_1$. Continuing in this fashion, he obtained sequences of (orthonormal) eigenfunctions u_j and corresponding eigenvalues λ_j, with $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$.

Although simple and beautiful, Dirichlet's principle (in its naïve form) is based on a false assumption, namely, that a minimiser $u \in V_g$ must exist because $J(v) \geq 0$ for all $v \in V_g$. This error was pointed out by Karl Theodore Wilhelm Weierstraß [7, pp. 390–391] in 1870, and the same objection applies to the variational arguments of Gauss, Liouville and Weber. During the period from 1870 to 1890, alternative existence proofs for the Dirichlet problem were devised by Hermann Amandus Schwarz, Carl Gottfried Neumann and Jules Henri Poincaré; see Gårding [25] and Kellogg [45, pp. 277–286]. We shall briefly describe the first of these proofs, Neumann's *Methode des arithmetischen Mittels*, after first introducing some important properties of the *double-layer potential*,

$$\mathrm{DL}\,\psi(x) = \int_\Gamma \psi(y) \frac{\partial}{\partial \nu_y} G(x, y)\,d\sigma_y \quad \text{for } x \notin \Gamma.$$

A surface potential of this type appears in the third Green identity (1.10), with $\psi = u|_\Gamma$; note the similarity with the general Poisson integral formula (1.11).

The double layer potential has a very simple form when the density is constant on Γ. In fact

$$\mathrm{DL}\,1(x) = \begin{cases} -1 & \text{for } x \in \Omega^-, \\ 0 & \text{for } x \in \Omega^+, \end{cases} \tag{1.15}$$

as one sees by taking $u = 1$ in (1.10) if $x \in \Omega^-$, and by applying the divergence theorem if $x \in \Omega^+$. Obviously, DL ψ is harmonic on $\Omega^\pm$, but the example $\psi = 1$ shows that the double-layer potential can have a jump discontinuity, and it turns