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978-0-521-66265-9 - Non-Perturbative Field Theory: From Two-Dimensional Conformal Field Theory to QCD in Four Dimensions

Yitzhak Frishman and Jacob Sonnenschein

Excerpt

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## PART I

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# Non-perturbative methods in two-dimensional field theory

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## 1

## From massless free scalar field to conformal field theories

In this chapter we analyze the simplest field theory, which is the theory of a free massless scalar field in two space-time dimensions, one space and one time.<sup>1</sup> The rich symmetry and algebraic structure of this theory encapsulates the basic concepts of two-dimensional conformal field theory, which will be the topic of the next chapter.

## 1.1 Complex geometry

It is convenient for the discussion of two-dimensional free scalar theory and later conformal field theories to introduce complex coordinates as follows:<sup>2</sup>

$$\xi = x^0 + ix^1 \quad \bar{\xi} = x^0 - ix^1. \quad (1.1)$$

We now take  $x^0$  and  $x^1$  to be in Euclidean space. Correspondingly we define the derivatives

$$\partial_\xi = \frac{1}{2}(\partial_0 - i\partial_1) \quad \partial_{\bar{\xi}} = \frac{1}{2}(\partial_0 + i\partial_1), \quad (1.2)$$

which is a special case of the decomposition to components of vectors, namely

$$\begin{aligned} A_\xi &= \frac{1}{2}(A^0 - iA^1) & A_{\bar{\xi}} &= \frac{1}{2}(A^0 + iA^1) \\ A^\xi &= (A^0 + iA^1) & A^{\bar{\xi}} &= (A^0 - iA^1). \end{aligned} \quad (1.3)$$

The metric of the flat Euclidean space-time  $ds^2 = dx^{0^2} + dx^{1^2}$  translates into  $ds^2 = d\xi d\bar{\xi}$ , namely

$$g_{\xi\bar{\xi}} = g_{\bar{\xi}\xi} = \frac{1}{2}, \quad g^{\xi\bar{\xi}} = g^{\bar{\xi}\xi} = 2, \quad g_{\xi\xi} = g_{\bar{\xi}\bar{\xi}} = g^{\xi\xi} = g^{\bar{\xi}\bar{\xi}} = 0. \quad (1.4)$$

With this metric at hand the scalar product of two vectors takes the form

$$A^\mu B_\mu = A^\xi B_\xi + A^{\bar{\xi}} B_{\bar{\xi}} = \frac{1}{2}(A^\xi B^{\bar{\xi}} + A^{\bar{\xi}} B^\xi). \quad (1.5)$$

Complex components of higher-order tensors relate in a similar manner to the real components, in particular for a symmetric two-tensor (like the

<sup>1</sup> The content of this chapter comprises the basics of massless scalar fields in two dimensions. This is covered in many textbooks.

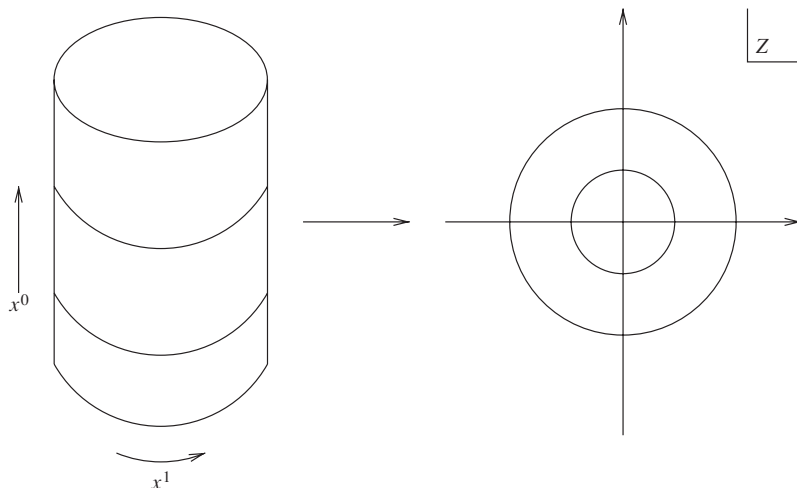
<sup>2</sup> The use of complex coordinates in the context of the bosonic string theory is described by Polyakov in [177].

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[More information](#)Fig. 1.1. The map between  $\xi$  and  $z$ .

energy-momentum tensor),

$$\begin{aligned}
 T &\equiv T_{\xi\xi} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \\
 \bar{T} &\equiv T_{\bar{\xi}\bar{\xi}} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \\
 T_{\xi\bar{\xi}} &= T_{\bar{\xi}\xi} = \frac{1}{4}(T_{00} + T_{11}).
 \end{aligned} \tag{1.6}$$

Often, especially in the context of string theory, the space direction is no longer  $\mathcal{R}$ , but rather is compactified on  $S^1$  so that  $x^1 \equiv x^1 + 2\pi$ . For such a geometry it is convenient to introduce the following conformal map:

$$\xi \rightarrow z = e^\xi = e^{x^0 + ix^1},$$

which maps the cylinder to the complex plane (see Fig. 1.1).

In particular the past  $x^0 = -\infty$  is mapped into the origin and the future  $x^0 = \infty$  into a circle with an infinite radius. It is clear that the relations between  $(\xi, \bar{\xi})$  and  $(x^0, x^1)$  derived above hold also between  $(z, \bar{z})$  and  $(\text{Real}(z), \text{Im}(z))$ . The holomorphic and anti-holomorphic derivatives with respect to  $z$  will be denoted by  $\partial \equiv \partial_z$  and  $\bar{\partial} \equiv \partial_{\bar{z}}$ .

## 1.2 Free massless scalar field

The action  $S$  of the free massless scalar field  $\hat{X}(z, \bar{z})$  is

$$\begin{aligned}
 S &= \int d^2x \mathcal{L} = \frac{1}{8\pi} \int d^2x \partial_\nu \hat{X} \bar{\partial}^\nu \hat{X} \\
 &= \frac{1}{4\pi} \int d^2\xi \partial_\xi \hat{X} \partial_{\bar{\xi}} \hat{X} = \frac{1}{4\pi} \int d^2z \partial \hat{X} \bar{\partial} \hat{X},
 \end{aligned} \tag{1.7}$$

## 1.3 Symmetries of the classical action

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where  $\mathcal{L}$  is the Lagrangian density. The factor  $\frac{1}{4\pi}$  is used to match the normalization of the bosonic string theory (with  $\alpha' = 2$ ). In the complex coordinate notation  $(\xi, \bar{\xi})$  and  $(z, \bar{z})$  the measure of the integral is  $d^2\xi = (i/2)d\xi \wedge d\bar{\xi}$  and  $d^2z = (i/2)dz \wedge d\bar{z}$ , respectively. Note that  $\mathcal{L}$  is a local expression and thus is the same for the Euclidean plane or for any compact two-surface.

Varying the scalar field  $\hat{X}(z, \bar{z}) \rightarrow \hat{X}(z, \bar{z}) + \delta\hat{X}(z, \bar{z})$  induces a variation in the action of the form

$$\delta S = -\frac{1}{2\pi} \int d^2z (\partial\bar{\partial}\hat{X})\delta\hat{X}. \quad (1.8)$$

The action is thus extremized by configurations that solve the corresponding equation of motion

$$\partial\bar{\partial}\hat{X} = 0. \quad (1.9)$$

It is thus clear that  $\partial\hat{X}$  is a holomorphic function and  $\bar{\partial}\hat{X}$  is an anti-holomorphic function, and the most general solution takes the form

$$\hat{X}(z, \bar{z}) = [X(z) + \bar{X}(\bar{z})]. \quad (1.10)$$

## 1.3 Symmetries of the classical action

By construction the action is invariant under translations and  $SO(2)$  rotations. Translations in  $x^0$  and  $x^1$  translate in complex coordinates to

$$z \rightarrow z + a; \quad \bar{z} \rightarrow \bar{z} + \bar{a}, \quad (1.11)$$

where  $a$  is a constant complex number, and the  $SO(2)$  rotations, in infinitesimal form, to

$$\delta z = -i\epsilon z; \quad \delta \bar{z} = i\epsilon \bar{z}, \quad (1.12)$$

where  $\epsilon$  is an infinitesimal real parameter.

When we go back to Minkowski space, the  $SO(2)$  rotations turn into  $SO(1, 1)$  transformations. In addition it is easy to realize that a shift of the field by a constant  $A$ ,

$$\hat{X}(z, \bar{z}) \rightarrow \hat{X}(z, \bar{z}) + A, \quad (1.13)$$

leaves the Lagrangian invariant. It is a special feature of two dimensions that the symmetry group of the action is in fact much richer since one can replace the constant  $A$  with  $A(z)$  and the constant  $\bar{A}$  with  $\bar{A}(\bar{z})$ , which are arbitrary holomorphic and anti-holomorphic functions, respectively,

$$\hat{X}(z, \bar{z}) \rightarrow \hat{X}(z, \bar{z}) + A(z); \quad \hat{X}(z, \bar{z}) \rightarrow \hat{X}(z, \bar{z}) + \bar{A}(\bar{z}). \quad (1.14)$$

These are the *affine current algebra transformations*.<sup>3</sup>

<sup>3</sup> Affine Lie algebras describing a physical system were first discussed in [27]. More references will be given in the next two chapters.

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In a similar manner the space-time translations (1.11) can also be elevated to holomorphic and anti-holomorphic transformations,

$$z \rightarrow f(z); \quad \bar{z} \rightarrow \bar{f}(\bar{z}), \quad (1.15)$$

referred to as two-dimensional *conformal transformations*. Affine current algebra transformations and conformal transformations will be further discussed in Sections 1.10 and 1.11

### 1.4 Mode expansion

The mode expansion of the classical solution depends on the boundary conditions. For the case where the underlying two-dimensional manifold is the infinite plane, a standard Fourier transform is used:

$$\hat{X}(x^0, x^1) = \int \frac{dk^1}{\sqrt{2\pi}\sqrt{k^0}} [a(k^1)e^{-ik \cdot x} + a^\dagger(k^1)e^{ik \cdot x}]. \quad (1.16)$$

If the range of the space coordinate is bounded, one may impose two types of boundary conditions, associated with closed and open strings. In the case of *closed strings* the boundary conditions

$$\hat{X}(x^0, x^1) = \hat{X}(x^0, x^1 + 2\pi) \quad (1.17)$$

are automatically obeyed by  $\hat{X}(z, \bar{z})$ . For this case the mode expansion is expressed in terms of a Laurent series,

$$\partial X = -i \sum_{n=-\infty}^{\infty} \frac{\alpha_n}{z^{n+1}} \quad \bar{\partial} \bar{X} = -i \sum_{n=-\infty}^{\infty} \frac{\bar{\alpha}_n}{\bar{z}^{n+1}}. \quad (1.18)$$

Integrating this expansion we get

$$\hat{X}(z, \bar{z}) = \mathcal{X} - i\mathcal{P} \ln(z\bar{z}) + i \sum_{m=-\infty, m \neq 0}^{\infty} \left( \frac{\alpha_m}{m} z^{-m} + \frac{\bar{\alpha}_m}{m} \bar{z}^{-m} \right), \quad (1.19)$$

with  $\mathcal{X}$  a constant and

$$\mathcal{P} = \alpha_0 = \bar{\alpha}_0. \quad (1.20)$$

For *open strings* the boundary conditions are of Neumann type, namely

$$\partial_1 \hat{X}(x^0, x^1 = 0) = \partial_1 \hat{X}(x^0, x^1 = \pi) = 0 \implies \partial \hat{X}(z, \bar{z} = z) = \bar{\partial} \hat{X}(z, \bar{z} = z). \quad (1.21)$$

The corresponding mode expansion takes the form

$$\hat{X}(z, \bar{z}) = \mathcal{X} - i\mathcal{P} \ln(z\bar{z}) + i \sum_{m=-\infty, m \neq 0}^{\infty} \frac{\alpha_m}{m} (z^{-m} + \bar{z}^{-m}). \quad (1.22)$$

**1.5 Noether currents and charges**

Associated with the symmetries (1.14) and (1.15) are conserved Noether currents and charges. In the Noether procedure one is instructed to elevate the global parameters of transformations into local ones and extract the associated currents from the variation of the action, namely  $\delta S \sim \int d^2x J_\mu \partial^\mu \epsilon$ . Let us apply this procedure first to the affine current algebra transformations so that we vary the action with respect to  $\delta \hat{X}(z, \bar{z}) = \epsilon(z, \bar{z})$  yielding

$$\delta S = \frac{1}{4\pi} \int d^2z [\partial \epsilon(z, \bar{z}) \bar{\partial} \hat{X}(z, \bar{z}) + \bar{\partial} \epsilon(z, \bar{z}) \partial \hat{X}(z, \bar{z})]. \quad (1.23)$$

Unlike the situation in more than two dimensions, and due to the fact that the symmetries (1.14) are in fact not only global ones but rather “half local”, the currents

$$J \equiv \partial X; \quad \bar{J} \equiv \bar{\partial} \bar{X} \quad (1.24)$$

are holomorphic and anti-holomorphic conserved,

$$\bar{\partial} J \equiv \bar{\partial} \partial X = 0; \quad \partial \bar{J} \equiv \partial \bar{\partial} \bar{X} = 0. \quad (1.25)$$

The classical currents are determined up to an overall constant.

A similar situation occurs with respect to the conformal transformation. Replacing in the infinitesimal version of (1.15)  $\delta z \rightarrow \epsilon(z, \bar{z})$  and  $\delta \bar{z} \rightarrow \bar{\epsilon}(z, \bar{z})$  one finds,

$$\delta S = \frac{1}{2\pi} \int d^2z [\partial \bar{\epsilon}(z, \bar{z}) \bar{\partial} \hat{X}(z, \bar{z}) \bar{\partial} \hat{X}(z, \bar{z}) + \bar{\partial} \epsilon(z, \bar{z}) \partial \hat{X}(z, \bar{z}) \partial \hat{X}(z, \bar{z})]. \quad (1.26)$$

The associated holomorphic and anti-holomorphic conserved energy-momentum tensor components are

$$T = -\frac{1}{2} \partial X \partial X; \quad \bar{T} = -\frac{1}{2} \bar{\partial} \bar{X} \bar{\partial} \bar{X}, \quad (1.27)$$

where the coefficients were chosen in a way that will turn out to be convenient when discussing the corresponding quantum generators.

**1.6 Canonical quantization**

Prior to imposing the canonical quantization condition one has to identify the time direction. There are several options. Using  $x^0$  as the time direction, the

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corresponding conjugate momentum of  $\hat{X}(z, \bar{z})$  is

$$\Pi = \frac{\delta \mathcal{L}}{\delta x_0 \hat{X}} = \frac{1}{4\pi} \partial_0 \hat{X},$$

and the standard quantization conditions are

$$\begin{aligned} [\hat{X}(x^0, x^1), \Pi(y^0, y^1)]_{x^0=y^0} &= i\delta(x^1 - y^1) \\ [\hat{X}(x^0, x^1), \hat{X}(y^0, y^1)]_{x^0=y^0} &= 0 \\ [\Pi(x^0, x^1), \Pi(y^0, y^1)]_{x^0=y^0} &= 0. \end{aligned} \quad (1.28)$$

These conditions yield the standard algebra of the creation and annihilation operators for (1.16),

$$[a(k^1), a^\dagger(p^1)] = \delta(k^1 - p^1); \quad [a(k^1), a(p^1)] = [a^\dagger(k^1), a^\dagger(p^1)] = 0. \quad (1.29)$$

Substituting the mode expansion (1.16) into the expressions of the Noether charges associated with the symmetries of the action (1.7) one finds that the energy-momentum operators are proportional to  $a^\dagger(k)a(k) + a(k)a^\dagger(k)$  and hence their vacuum expectation values are proportional to  $\delta(0) \sim L$ , where  $L$  is the size of the space direction. It is thus clear that for the infinite Euclidean plane (or a Minkowski space-time with space  $\mathcal{R}$ ) these expectation values diverge. One then defines the **normal ordered** operators:

$$:\mathcal{O}: \equiv \mathcal{O} - \langle 0|\mathcal{O}|0\rangle. \quad (1.30)$$

For free fields this is equivalent to ordering annihilation operators to the right of creation operators, and sufficient to make  $:\mathcal{O}:$  finite.

Using the algebra of the creation and annihilation operators and the normal ordered Hamiltonian, the construction of the Fock space is standard. One defines the vacuum state  $|0\rangle$  such that

$$a(k^1)|0\rangle = 0. \quad (1.31)$$

The states in the Fock space are

$$\prod_i a^\dagger(k_i)^{n_i} |0\rangle, \quad (1.32)$$

and their energies, by applying the Hamiltonian,

$$H|\prod_i a^\dagger(k_i)^{n_i} |0\rangle = \sum_i (k_i^0) n_i(k_i) \prod_i a^\dagger(k_i)^{n_i} |0\rangle. \quad (1.33)$$

The canonical quantization for the scalar field on a compact space direction, with the boundary conditions of open or closed string, (1.21) and (1.17), respectively, follows very similar steps. Imposing the quantization conditions (1.28) above implies the following algebra for the  $\alpha_n$  operators of the open string and

## 1.7 Radial quantization

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for the  $\alpha_n$  and  $\bar{\alpha}_n$  operators for the closed string:

$$\begin{aligned} [\alpha_m, \alpha_n] &= m\delta_{m+n} \\ [\bar{\alpha}_m, \bar{\alpha}_n] &= m\delta_{m+n} \\ [\alpha_m, \bar{\alpha}_n] &= 0. \end{aligned} \quad (1.34)$$

It is thus clear that  $\alpha_n$  operators are related to the  $a(k)$  operators as

$$\alpha_m = \sqrt{m}a(m), m > 0; \quad \alpha_{-m} = \sqrt{m}a^\dagger(m), m > 0. \quad (1.35)$$

## 1.7 Radial quantization

For the case of a cylinder-like two-dimensional manifold, namely, where the space direction is compactified so that  $x^1 \equiv x^1 + 2\pi$ , it is natural to use the  $z = e^{x^0 + ix^1}$  coordinates. Space translations  $x^1 \rightarrow x^1 + a$  take the form of multiplying by a phase factor  $z \rightarrow e^{ia}z$ , and time translations  $x^0 \rightarrow x^0 + a$  turn into dilatations  $z \rightarrow e^a z$ . Rotations  $(x^0 + ix^1) \rightarrow (c + is)(x^0 + ix^1)$ , go into  $z \rightarrow z^{(c+is)}$ , with  $(c + is) = e^{i\theta}$ ,  $\theta$  the rotation angle. Correspondingly the generators of these transformations change their geometrical operation. For instance the Hamiltonian obviously goes into the dilatation generator. Moreover, generators which are Noether charges transform into contour integrals. Recall that the Noether charge is  $Q = \int dx^1 J_0(x^1)$  which in the new coordinates reads  $Q = \int d\theta J_r(\theta)$  so that we can write,

$$Q = \frac{1}{2\pi i} \oint [dz J(z) + d\bar{z} \bar{J}(\bar{z})], \quad (1.36)$$

where the contour integral is performed at some radius and the sign convention we adopt is that both the  $dz$  and  $d\bar{z}$  integral are taken to be positive for the counter-clockwise sense.

The infinitesimal transformation of an operator generated by the Noether charge  $Q$  is given by:

$$\delta_{\epsilon, \bar{\epsilon}} \mathcal{O} = \frac{1}{2\pi i} \oint [dz J(z) \epsilon(z), \mathcal{O}(w, \bar{w})] + d\bar{z} [\bar{J}(\bar{z}) \bar{\epsilon}(\bar{z}), \mathcal{O}(w, \bar{w})]. \quad (1.37)$$

Define a product  $R$  of two operators  $A(z)B(w)$  as taken radially, namely<sup>4</sup>

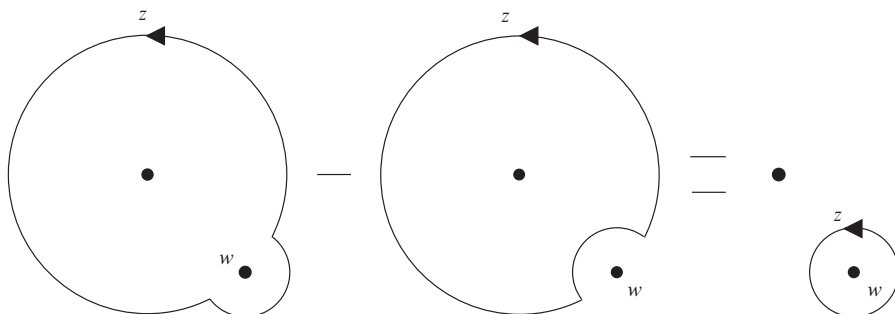
$$R(A(z)B(w)) = A(z)B(w), |z| > |w|; \quad B(w)A(z), |w| > |z|. \quad (1.38)$$

In Fig. 1.2 we show the two contour integrals that lead to a contour integral around  $w$ , the location of the operator  $\mathcal{O}$ , so that the infinitesimal transformation is given by,

$$\delta_{\epsilon, \bar{\epsilon}} \mathcal{O} = \frac{1}{2\pi i} \oint [dz \epsilon(z) R(J(z) \mathcal{O}(w, \bar{w})) + d\bar{z} \bar{\epsilon}(\bar{z}) R(\bar{J}(\bar{z}) \mathcal{O}(w, \bar{w}))]. \quad (1.39)$$

<sup>4</sup> The notion of radial quantization was introduced in [104]. This construction was used in the context of complex geometry in [93].



Fig. 1.2. A contour around  $w$  from the commutator.

We now apply this formulation to the symmetry generators (discussed in Section 2.1):

- (i) The infinitesimal affine current algebra transformation  $\hat{X}(z, \bar{z}) \rightarrow \hat{X}(z, \bar{z}) - \epsilon(z)$  is generated by the holomorphic current  $J(z) = \partial X$  via

$$\begin{aligned} \delta_\epsilon \hat{X}(w, \bar{w}) &= \frac{1}{2\pi i} \oint dz \epsilon(z) R(\partial X(z) \hat{X}(w, \bar{w})) \\ &= \frac{1}{2\pi i} \oint dz \frac{-1}{z-w} \epsilon(z) = -\epsilon(w), \end{aligned} \quad (1.40)$$

where we have used for the product of operators,

$$R(X(z)X(w)) = -\log(z-w) + \text{finite terms}. \quad (1.41)$$

This is an example of the concept of operator product expansion, which is addressed in the next section.

- (ii) In a similar manner we can compute the transformation of  $\partial X$  generated by the energy momentum tensor  $T$

$$\begin{aligned} \delta_\epsilon \partial X(w) &= \frac{1}{2\pi i} \oint dz \epsilon(z) R\left(-\frac{1}{2} : \partial X(z) \partial X(z) : \partial X(w)\right) \\ &= \frac{1}{2\pi i} \oint dz \frac{1}{(z-w)^2} \partial X(z) \epsilon(z) = \partial \epsilon(w) \partial X(w) + \epsilon(w) \partial^2 X(w), \end{aligned} \quad (1.42)$$

which is indeed the infinitesimal transformation of the holomorphic current  $J = \partial X(z)$ . The generator  $T$  is normal ordered using the following expression:

$$T(w) = -\frac{1}{2} : \partial X(z) \partial X(w) : \equiv -\frac{1}{2} \lim_{z \rightarrow w} \left[ \partial X(z) \partial X(w) + \frac{1}{(z-w)^2} \right]. \quad (1.43)$$

**1.8 Operator product expansion**

In computing the contour integrals associated with infinitesimal transformations we have made use of the operator product expansions of pairs of operators.<sup>5</sup> The singularities that occur when the points are taken to approach one another are captured in the notion of *operator product expansion (OPE)*,

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k c_{ij}^k(x-y)\mathcal{O}_k(y), \quad (1.44)$$

where  $c_{ij}^k(x-y)$  are the *coefficient functions* which are singular in the limit of  $x \rightarrow y$ . Such expansions were proven to hold in renormalizable field theories. The OPEs are an essential tool in exploring quantum field theories. Recall that all of the information on the QFT is encoded into the values of all possible correlation functions of the complete set of local

operators  $\mathcal{O}_i(x)$ , namely,  $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$ . In particular, one is interested in the behavior of these correlation functions when two or more points approach each other, which is encapsulated in the OPEs. For all applications discussed here the OPEs are treated as asymptotic expansions and only their singular terms will be specified. For the present case of two-dimensional free massless scalar field theory the OPE converges and in fact, as will be discussed in Section 3.7.2, a similar situation occurs in all 2d CFTs.

The OPEs of the free massless scalar can be deduced from its propagator, which can be evaluated from the solution. It takes the form:

$$\langle \hat{X}(z\bar{z})\hat{X}(w\bar{w}) \rangle = -\log|z-w|^2. \quad (1.45)$$

In terms of the separation of the solution into holomorphic and anti-holomorphic parts the two propagators read:

$$\langle X(z)X(w) \rangle = -\log(z-w); \quad \langle \bar{X}(\bar{z})\bar{X}(\bar{w}) \rangle = -\log(\bar{z}-\bar{w}). \quad (1.46)$$

By differentiating the last relation with respect to  $z$  and to  $w$  one finds the short distance expansion of other operators like  $J(z)$ ,  $T(z)$  etc. In particular the OPE of the currents is

$$J(z)J(w) = \partial X(z)\partial X(w) = -\frac{1}{(z-w)^2} + \text{finite terms}, \quad (1.47)$$

with a similar result for the anti-holomorphic currents.

A different, though equivalent, approach is to write the OPE as a Taylor expansion in  $(z-w)$  and  $(\bar{z}-\bar{w})$  in the following form:

$$\begin{aligned} \hat{X}(z, \bar{z})\hat{X}(w\bar{w}) &= -\log|z-w|^2 + \sum_{k=1}^{\infty} \frac{1}{k!} [(z-w)^k : (\partial^k \hat{X}(w, \bar{w})) \hat{X}(w, \bar{w}) : \\ &\quad + (\bar{z}-\bar{w})^k : (\bar{\partial}^k \hat{X}(w, \bar{w})) \hat{X}(w, \bar{w}) :]. \end{aligned} \quad (1.48)$$

<sup>5</sup> Wilson introduced for the first time the concept of an operator product expansion [219]. It was used for two-dimensional conformal field theories in [33].