S. Morosawa, Y. Nishimura, M. Taniguchi, T. Ueda
Kochi University, Osaka Medical College, Kyoto University, Kyoto University

Holomorphic Dynamics

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1

Dynamics of polynomial maps

A simple mechanism can make a chaos. The case of iteration of polynomial maps is no exception. Indeed an extremely simple map, $z \to z^2 + c$, can create deep confusion. Understanding of, or, more modestly, an attempt to understand, such phenomena is the theme of this book.

More precisely, consider a monic polynomial

$$P(z) = z^k + a_1 z^{k-1} + \cdots + a_{k-1} z + a_k$$

of degree k. Then P can be considered as a self-map of $\mathbb{C}$, and the issue is to discuss what happens when we apply P to a given point z iteratively. Set

$$z_0 = z, z_1 = P(z), \ldots, z_n = P(z_{n-1}), \ldots,$$

and consider the behavior of the sequence $\{z_n\}_{n=1}^{\infty}$ or the orbit of z. Here, one might take some stable dependence on the initial value z for granted. But this is not true in general, and the set of unstable initial values is, almost always, incredibly complicated, and nowadays is called a fractal. Moreover, for almost every such initial value, the behavior is chaotic enough.

First, in §1.1, we give several typical examples of fractal sets caused by polynomial maps, and explain basic properties of them. Then §1.2 summarizes local theory near a fixed point of a general holomorphic map.

Turning to quadratic polynomial maps as above, we introduce the Mandelbrot set, which in itself is a fractal and represents the very interesting family of quadratic polynomial maps, in §1.3.

As the fundamental tools to measure how complicated the set considered is, we introduce in §1.4 the Hausdorff measures and the logarithmic capacity, and give some basic facts about them.

Finally, in §1.5, we define a polynomial-like map, which is one of the fundamental concepts in the modern theory of complex dynamics, and explain some basic results.

In this chapter, we explain not only basic facts on the dynamics of polynomials, but also several central issues about complex dynamics. In particular, we include many important results without proofs. As general references for those results, we cite books by Beardon (1991), Carleson and Gamelin (1993), Milnor (1990), and Steinmetz (1993).

1.1 The set of escaping points for a polynomial

1.1.1 *Typical examples of polynomials*

Fix a monic polynomial

$$P(z) = z^k + a_1 z^{k-1} + \cdots + a_{k-1} z + a_k$$

of degree $k \geq 2$, and consider the sequence of $\{P^n\}_{n=0}^{\infty}$, where and in what follows, we set

$$P^0(z) = z, P^1(z) = P(z), \ldots, P^n(z) = P(P^{n-1}(z)).$$

Then for every z with sufficiently large $|z|$, we have

$$|z| < |P(z)| < |P^2(z)| < \cdots.$$

In fact, choose an $R > 2$ so large that $|P(z)| > |z|^k/2$ for every z with $|z| > R$. Then inductively, we see that

$$R < |z| < |z|^k/2 < |P(z)| < \cdots < |P^{n-1}(z)| < |P^{n-1}(z)|^k/2 < |P^n(z)|.$$

Thus for every such z, $P^n(z)$ tends to ∞ in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition We set

$$I_P = \left\{ z \in \mathbb{C} \;\middle|\; \lim_{n \to \infty} P^n(z) = \infty \right\},$$

and call it the *set of escaping points* for P. Also we set

$$K_P = \mathbb{C} - I_P$$

and call it the *filled Julia set* of P. The boundary $J_P = \partial K_P$ of K_P is called the *Julia set* of P.

Remark Since $k \geq 2$, any polynomial

$$P(z) = a_0 z^k + a_1 z^{k-1} + \cdots + a_{k-1} z + a_k \quad (a_0 \neq 0)$$

is affine conjugate to a monic polynomial of degree k. More precisely, set $A(z) = \omega z$, where ω is a $(k-1)$th root of a_0, and we see that $A \circ P \circ A^{-1}(z)$ is a monic polynomial of degree k.

The following example is clear.

Example 1.1.1 *Let* $P(z) = z^2$*. Then we have*

$$I_P = \{|z| > 1\}, \quad K_P = \{|z| \leq 1\}, \quad J_P = \{|z| = 1\}.$$

Example 1.1.2 *Let* $P(z) = z^2 - 2$*. Then we have*

$$I_P = \mathbb{C} - [-2, 2], \quad K_P = J_P = [-2, 2],$$

where $[-2, 2]$ *denotes the interval* $\{x \in \mathbb{R} \mid -2 \leq x \leq 2\}$.

Verification Set $B(z) = z + 1/z$, and we can take the single valued branch of B^{-1} from $\mathbb{C} - [-2, 2]$ onto $\{|z| > 1\}$. A simple calculation shows that $P(z) = B \circ P_0 \circ B^{-1}(z)$ with $P_0(z) = z^2$. Hence by Example 1.1.1, we see that I_P contains $\mathbb{C} - [-2, 2]$. On the other hand, it is clear that $P([-2, 2]) = [-2, 2]$, and hence the assertions follow.

Recall that P restricted to $[-2, 2]$ is a typical unimodal map which represents an action that expands the interval to twice its length and then folds it into two.

In general, J_P has a very complicated shape. Actually, the above examples are exceptions. We show the variety of the Julia sets by examples.

Remark The reasons why the Julia sets in the examples below have such shape are not easy, and need several results discussed later. But for the sake of convenience, we include brief explanations.

We start by clarifying what we consider as fractal sets. There are several ways to define fractal sets. Self-similarity relates closely to fractal sets. But there exist so many complicated sets, for which self-similarity is not so clear or seems to be absent. From a quantitative viewpoint, we may call every simple curve with Hausdorff dimension, defined in §1.4, greater than 1 a fractal curve. But we take a looser definition (also see §4.1).

Definition Let E be a closed subset of $\mathbb{C}$ without interior. We say that E is a *fractal set* if E cannot be represented as a countable union of rectifiable curves.

We say that a domain D in $\widehat{\mathbb{C}}$ is a *fractal domain* if the boundary ∂D is a fractal set.

As to fractal sets in the multi-dimensional complex number space, we will discuss these in Chapters 6–9.

Even in the case that the Julia set J_P is a simple closed curve, J_P is terribly complicated except for trivial cases.

To give such examples, we perturb $P(z) = z^2$ in Example 1.1.1. That is, consider $P_c(z) = z^2 + c$ with sufficiently small c. Then simple computations show that there is a fixed point z_c, a solution of $P(z) = z$, near $z = 0$ and hence the value of the derivative at z_c is also near to 0. Moreover, we will show that the structure of J_{P_c} is circle-like.

Example 1.1.3 *When $|c|$ is sufficiently small, the Julia set J_{P_c} of $P_c(z) = z^2+c$ is a simple closed curve, but has a very complicated shape, called a fractal circle. See Figure 1.1(a), the Julia set of $z^2 + c$, $c = 0.59\cdots + i0.43\cdots$, and Figure 1.1(b), that of $z^2 + c$, $c = 0.33\cdots + i0.07\cdots$.*

Verification By the corollary to Theorem 1.5.1, each Julia set is a quasicircle. In particular, it is a simple closed curve. Theorem 1.4.7 implies that it is a fractal set.

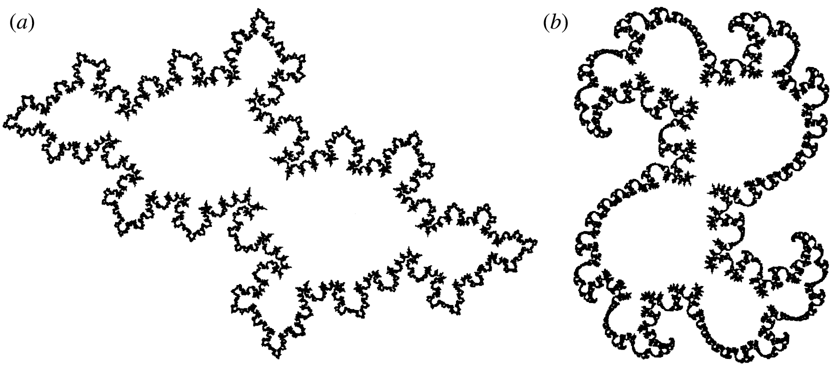

Fig. 1.1. Jordan curves as Julia sets

Definition We say that a compact set K in $\widehat{\mathbb{C}}$ is *locally connected* if, for every $z_0 \in K$ and every neighborhood U of z_0, $U \cap K$ contains a connected neighborhood of z_0 in K.

A connected and locally connected compact set is arcwise connected. For some conditions on local connectedness, see §4.4.

Example 1.1.4 *Set*

$$E = \left\{ x + iy \,\middle|\, 0 < x < 1, y = \sin \frac{1}{x} \right\},$$

and the closure of E is not locally connected.

Definition We call a compact set K a *dendrite* if K is a connected and locally connected compact set without interior whose complement $\mathbb{C} - K$ is connected.

Example 1.1.5 *A typical example of a dendrite is the (filled) Julia set of $z^2 + i$. As other examples, we cite in Figure 1.2 the (filled) Julia sets of*

(*a*) $z^2 + c, \quad c \in \mathbb{R}, \ c^3 + 2c^2 + 2c + 2 = 0,$
(*b*) $z^3 + c, \quad c = \sqrt{\omega - 1}, \ \omega^2 + \omega + 1 = 0.$

Verification These Julia sets are connected by Theorem 1.1.4, for the orbit of the unique critical value 0 is preperiodic and hence bounded. They have no interior points, which can be shown as in the proof of Theorem 4.2.18. Further, since they are subhyperbolic, defined in §4.4, the Julia sets are locally connected.

When the Julia set is a dendrite, it coincides with the filled Julia set. There is another typical case that these are coincident with each other. It is the case that the Julia set of a polynomial is a Cantor set. We will define and discuss Cantor sets in §1.4. Here we temporarily call a set E a Cantor set if E has no isolated points, and every connected component of E consists of a single point. Then it is easy to see that such an E has uncountably many connected components.

By definition, if a closed set without interior has uncountably many connected components, then it is a fractal set. We give in Figure 1.3 the Julia sets of

(*a*) $z^2 + c, \quad c = 0.38 \cdots + i0.22 \cdots,$
(*b*) $z^2 + c, \quad c = -0.82 \cdots + i0.33 \cdots.$

These Julia sets are Cantor sets by Theorem 1.1.6, for the c are outside of the Mandelbrot set defined in §1.3.

In the case of polynomials P of degree greater than 2, the filled Julia set may have interior points even if it has uncountably many connected components. We give the following example.

Example 1.1.6 *The Julia set of $P(z) = z^3 + az$ with $a = 0.038 + i1.95$ consists of uncountably many components and countably many components have interior points. See Figure 1.4.*

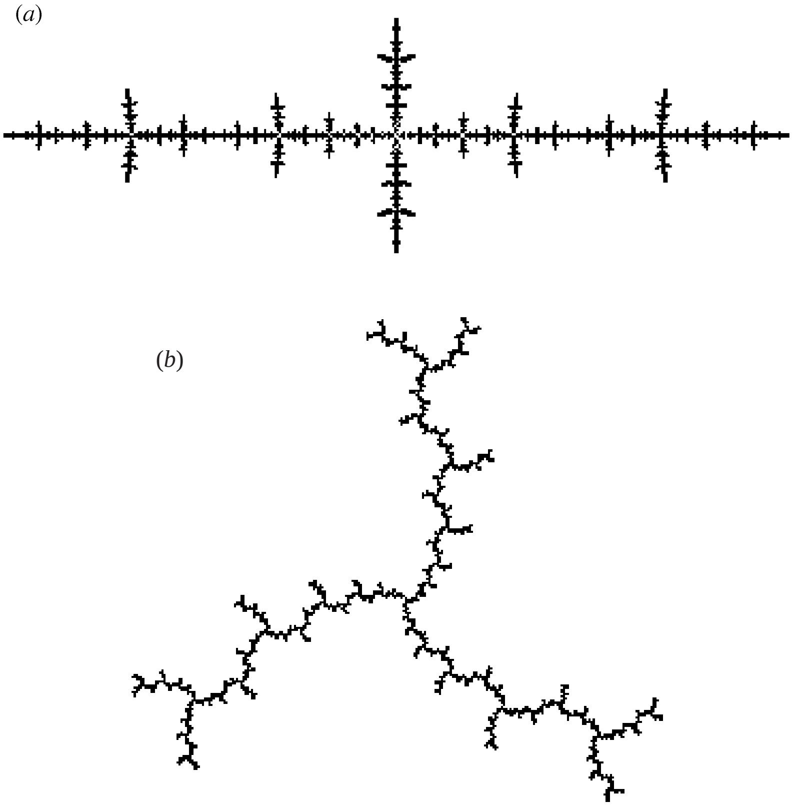

Fig. 1.2. Dendrites as Julia sets

Verification Since Theorem 2.3.6 shows that Julia sets are perfect sets, a disconnected Julia set consists of uncountably many components. To see the filled Julia set of P has countably many connected components which have interior points, we need only pursue the orbit of the finite critical points $\pm\sqrt{-a/3}$, as is seen from the fact proved in Chapter 2. We remark that the shape of each component depends on the polynomial induced by a suitable renormalization, which is defined in §1.5.

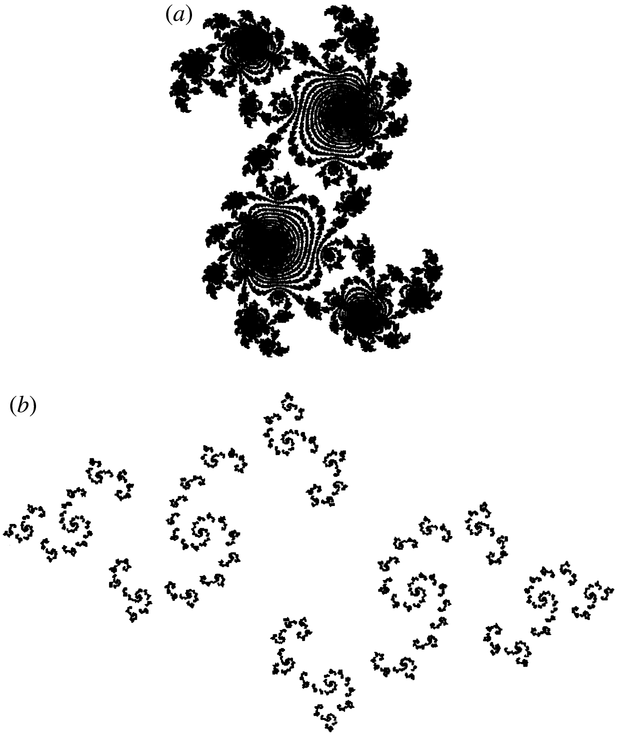

Fig. 1.3. Cantor sets as Julia sets

1.1.2 Basic properties of the filled Julia set

We have seen that the Julia set may have incredibly varied shapes even restricted to simple polynomial maps. One purpose of this book is to discuss such variety of shapes and characters of the Julia sets, though we stress explanation of the unified nature which a fairly large class of complex dynamics are equipped with.

We start with the following elementary fact.

Theorem 1.1.1 *Let $P(z)$ be a polynomial of degree not less than 2. Then the filled Julia set K_P is a non-empty compact set.*

Moreover, I_P is a domain, that is a connected open set.

Remark When $P(z) = z + c$ ($c \neq 0$), it is clear that K_P is empty. The case of a general linear map is left to the reader.

Fig. 1.4. Disconnected Julia set with non-empty interior

Proof of theorem Since the degree of P is not less than 2, P always has a fixed point, which belongs to K_P by definition. Hence K_P is non-empty.

Next, as noted before, setting $V = \{z \in \mathbb{C} \mid |z| > R\}$ for a sufficiently large R, we have $P(V) \subset V$ and $V \subset I_P$. On the other hand, for every $z_0 \in I_P$, the orbit elements $P^n(z_0)$ tend to ∞, and hence there is an N such that $P^N(z_0) \in V$. Thus we conclude that

$$I_P = \bigcup_{n=1}^{\infty} (P^n)^{-1}(V), \tag{1.1}$$

which in particular implies that I_P is an open set and that K_P is a bounded closed set.

Finally, if there exists a connected component of $(P^n)^{-1}(V)$ disjoint from V, then it should be bounded, for $(P^n)^{-1}(V)$ contains V. This contradicts the maximal principle, and hence we conclude that $(P^n)^{-1}(V)$ is a domain containing V. Thus the above equation shows the assertion. □

The equation (1.1) also implies the following.

Theorem 1.1.2 (Complete invariance) *Let $P(z)$ be a polynomial of degree not less than 2. Then the sets I_P, K_P and J_P are completely invariant under the action by P, i.e., letting E be one of these sets, we have*

$$P(E) = P^{-1}(E) = E.$$

Proof By (1.1), we see that $P^{-1}(I_P) = I_P$. Since $P((P^n)^{-1}(V)) = (P^{n-1})^{-1}(V)$, we also have that $P(I_P) = I_P$. Thus I_P is completely invariant, and so is the complement K_P. Since P is a continuous open map, J_P is also completely invariant. □

Theorem 1.1.2 implies that every point of K_P has a bounded orbit. Hence we can characterize K_P as the set of points with bounded orbits.

Definition A closed set E in $\widehat{\mathbb{C}}$ is called *perfect* if E has no isolated points.

A domain D in $\widehat{\mathbb{C}}$ is *simply connected* if the complement $\widehat{\mathbb{C}} - D$ is connected.

Theorem 1.1.3 *Let $P(z)$ be a polynomial of degree not less than* 2*. Then the filled Julia set K_P is a perfect set.*

Moreover, every connected component of the interior of K_P is simply connected.

Proof Suppose that there is an isolated point z_0 of K_P. Then we can draw a simple closed curve C in I_P such that the intersection of the interior W of C in $\mathbb{C}$ and K_P consists only of the point z_0.

Now take such a V as in the proof of Theorem 1.1.1. Then by (1.1), we can find N such that $P^N(C) \subset V$. Since $P^N(z_0)$ is not contained in V, $P^N(W)$ should contain $\mathbb{C} - V$. Then complete invariance of K_P means that $K_P = \{z_0\}$. In particular, $P^{-1}(z_0) = \{z_0\}$, and hence $P(z)$ should have the form $a(z - z_0)^k + z_0$. But then z_0 should belong to the interior of K_P, which is a contradiction.

Next, if there were a connected component of the interior of K_P which were not simply connected, then $I_P \cup J_P = \overline{I_P}$ would not be connected, which contradicts Theorem 1.1.1. □

Remark For every $z \in I_P$, the closure of $\bigcup_{n=1}^{\infty}(P^n)^{-1}(z)$ contains J_P, and for every $z \in J_P$, we have

$$\overline{\bigcup_{n=1}^{\infty}(P^n)^{-1}(z)} = J_P.$$

These facts follows from the non-normality of $\{P^n\}$ on the Julia set (see Theorem 2.3.4), and give one of the standard ways to draw the Julia set.

Definition The solutions of $P'(z) = 0$ are called *critical points* of $P(z)$. We denote by C_P the set of all critical points of $P(z)$, and call it the *critical set* of

$P(z)$:

$$C_P = \{z \in \mathbb{C} \mid P'(z) = 0\}.$$

Now whether $I_P \cup \{\infty\}$ is simply connected or not depends on the location of the critical set.

Theorem 1.1.4 *Let $P(z)$ be a polynomial of degree not less than 2. Then the filled Julia set K_P is connected if and only if $I_P \cap C_P = \emptyset$.*

Proof As in the proof of Theorem 1.1.1, fix R so that $V = \{|z| > R\}$ is contained in I_P.

First, suppose that $I_P \cap C_P$ is the empty set. Then P^n gives a smooth k^n-sheeted covering of $V_n = (P^n)^{-1}(V)$ onto V, where k is the degree of P. Since $V \cup \{\infty\}$ is simply connected, so is every $V_n \cup \{\infty\}$. Hence the union $I_P \cup \{\infty\}$ of increasing domains $V_n \cup \{\infty\}$ is simply connected.

Next suppose that $I_P \cap C_P$ is not empty, and let N be the minimum of n such that $V_n \cap C_P \neq \emptyset$. Then applying Lemma 1.1.5 below to the proper holomorphic map $P : (V_N - \overline{V_{N-1}}) \to (V_{N-1} - \overline{V_{N-2}})$, we see that V_N is not simply connected. Hence K_P is not connected. □

Lemma 1.1.5 (Riemann–Hurwitz formula for domains) *Let D_1 and D_2 be domains in $\widehat{\mathbb{C}}$ whose boundaries consist of a finite number of simple closed curves. Let $f(z)$ be a proper holomorphic map of D_1 onto D_2. Then:*

(i) *Every $z \in D_2$ has the same number k of preimages including multiplicity.*

(ii) *Denote by N the number of critical points of f in D_1 including multiplicity. Then*

$$(2 - d_1) = k(2 - d_2) - N,$$

where d_j is the number the boundary components of D_j.

In particular, when both the D_j are simply connected, f has at most $k - 1$ critical points.

The number k in the lemma is called the degree of f.

Proof The first assertion follows since f is a proper open map. The second assertion follows by applying Euler's formula to suitable triangulations of D_j. □

A typical example of disconnected K_P is those such as in Figure 1.3. We say that a compact set E is *totally disconnected* if every connected component of E consists of a single point.

Theorem 1.1.6 *Let $P(z)$ be a polynomial of degree not less than 2. If the set I_P of escaping points contains C_P, then K_P is totally disconnected. In particular, $K_P = J_P$.*

Proof The assumption implies that the forward orbit

$$C^+(P) = \bigcup_{n=1}^{\infty} P^n(C_P)$$

of C_P accumulates only to ∞. Hence there is a simple closed curve C such that the interior W of C contains K_P and the exterior $V = \mathbb{C} - \overline{W}$ contains $C^+(P)$. As in the proof of Theorem 1.1.1, we see that $I_P = \bigcup_{n=1}^{\infty}(P^n)^{-1}(V)$, and hence $P^N(V \cup C_P) \subset V$ for a sufficiently large N. This implies that $(P^N)^{-1}$ has k^N single valued holomorphic branches on W, which we denote by $\{h_j\}_{j=1}^{k^N}$. Then the elements of $\mathcal{U}_1 = \{U_j = h_j(W)\}$ are pairwise disjoint, and each U_j contains k^N images $\mathcal{U}_2 = \{U_{jk} = h_j(U_k)\}_{k=1}^{k^N}$ of them. Inductively, we can define the set $\mathcal{U}_\ell$ of domains for every positive integer ℓ, and $K_P = \bigcap_{\ell=1}^{\infty} \mathcal{U}_\ell$.

Thus it suffices to show that, for any sequence of domains $U(\ell) \in \mathcal{U}_\ell$ such that $U(\ell) \subset U(\ell - 1)$, the intersection $E = \bigcap_{\ell=1}^{\infty} U(\ell)$ consists of a single point. For this purpose, let $R_j = \{1 < |z| < m_j\}$ be the ring domain conformally equivalent to $W - \overline{U_j}$ for every j. We call the quantity $\log m_j$ the *modulus* of R_j. Set $m = \min_j m_j$. Then since every $U(\ell - 1) - \overline{U(\ell)}$ is conformally equivalent to one of the R_j, it contains a ring domain conformally equivalent to $\{1 < |z| < m\}$.

Here the crucial fact is the following subadditivity of the moduli of ring domains. Suppose that a ring domain R contains disjoint ring domains S_1 and S_2 essentially, i.e., each S_j separates $\widehat{\mathbb{C}} - R$. Let $\log m$ and $\log m_j$ be the moduli of R and S_j. Then

$$\log m \geq \log m_1 + \log m_2.$$

This is a direct consequence of the extremal length characterization of the modulus.

Thus $W - \overline{U(\ell)}$ contains essentially a ring domain with modulus $\ell \log m$. Hence $W - E$ contains essentially a ring domain with arbitrarily large modulus. If E contains more than two points, this is impossible. □

Remark In the case that I_P contains C_P, the proof of Theorem 1.1.6 indicates how to describe the dynamics on $K_P = J_P$ symbolically.

Take the *space of sequences of k symbols*

$$\Sigma_k = \{0, 1, \ldots, k-1\}^{\mathbb{Z}^+} = \{(m_0, m_1, \ldots) \mid m_j \in \{0, 1, \ldots, k-1\}\},$$

where $\mathbb{Z}^+$ is the set of non-negative integers. We equip Σ_k with the product topology. In other words, we take the topology so that the injection

$$(m_0, m_1, \ldots) \to \sum_{j=0}^{\infty} 2m_j(2k-1)^{-j-1}$$

becomes a homeomorphism into $\mathbb{R}$.

Then letting k be the degree of P, we can identify J_P with Σ_k so that the dynamics of P on J_P equals the canonical *shift operator*

$$\sigma((m_0, m_1, \ldots)) = (m_1, m_2, \ldots)$$

on Σ_k. Further see §3.2.1 and §7.4.

1.2 Local behavior near a fixed point

1.2.1 Schröder equation for λ *with* $|\lambda| < 1$

The behavior of the dynamics of a polynomial $P(z)$ can be fairly well understood at least near a fixed point of $P(z)$. We gather in this subsection classical and basic facts on local behavior of a general holomorphic function near a fixed point. First, ∞ is a fixed point of $P(z)$ considered as a holomorphic endomorphism of $\widehat{\mathbb{C}}$, and has a distinguished character.

Theorem 1.2.1 (Böttcher) *Fix a monic polynomial $P(z)$ of degree $k \geq 2$. Then for a sufficiently large R, there exists a conformal map $\phi(z)$ of $V = \{|z| > R\}$ into $\mathbb{C}$ which has a form*

$$\phi(z) = bz + b_0 + \frac{b_1}{z} + \cdots$$

and satisfies

$$\phi(P(z)) = \{\phi(z)\}^k. \tag{1.2}$$

Moreover, such a $\phi(z)$ is uniquely determined up to multiplication by a $(k-1)$th root of 1.

We call such a function $\phi(z)$ as in the above theorem a *Böttcher function* for P at ∞. Also see §7.3.

Proof We may assume that $P(V) \subset V$, and hence there exists a single valued holomorphic branch $\psi(z)$ of

$$\log \frac{P(z)}{z^k}$$

such that $\lim_{z\to\infty} \psi(z) = 0$. Since $P(z) = z^k \exp \psi(z)$, we see inductively that

$$P^n(z) = z^{k^n} \exp\{k^{n-1}\psi(z) + \cdots + \psi(P^{n-1}(z))\}.$$

Hence we can take

$$\phi_n(z) = z \exp\left\{\frac{1}{k}\psi(z) + \cdots + \frac{1}{k^n}\psi(P^{n-1}(z))\right\}$$

as a branch of the k^nth root of $P^n(z)$.

Now we may assume that $\psi(z)$ is bounded on V. Then

$$\sum_{j=1}^{\infty} k^{-j}\psi(P^{j-1}(z))$$

converges uniformly on V, and hence so does the sequence $\{\phi_n(z)\}$. The limit $\phi(z)$ of $\phi_n(z)$ is holomorphic on V and satisfies equation (1.2), for $\phi_n(P(z)) = \{\phi_{n+1}(z)\}^k$. Since $\phi(z)$ is injective near ∞, we conclude with the first assertion. The second assertion follows from (1.2). □

In general, a Böttcher function cannot be continued analytically to the whole of I_P. Also see the remark below. But we have the following.

Proposition 1.2.2 *Let $\phi(z)$ be a Böttcher function for a monic polynomial $P(z)$ of degree $k \geq 2$, and set*

$$g(z) = \log|\phi(z)|.$$

Then $g(z)$ can be extended to a harmonic function on the whole of I_P.

Moreover, set $g(z) = 0$ for every $z \in K_P$. Then $g(z)$ is a continuous subharmonic function on $\mathbb{C}$.

The function $g(z)$ in the proposition is the Green function on $I_P \cup \{\infty\}$ with pole ∞, which will be defined in §1.4.3.

Proof Since $I_P = \bigcup_{n=1}^{\infty}(P^n)^{-1}(V)$, we extend $g(z)$ to the whole of I_P by setting

$$g(z) = \frac{1}{k^n}g(P^n(z))$$

if $z \in (P^n)^{-1}(V)$. This $g(z)$ is well defined by (1.2) and clearly harmonic.

Next, set $A = \max_{z \in \overline{V} - P(V)} g(z)$, and take any neighborhood W of K_P. Then

$$(P^N)^{-1}(V) \cap W = \varnothing$$

implies that

$$g(z) \leq k^{-N} A \qquad (z \in W).$$

Hence we have the second assertion. □

Remark It is known as Sibony's theorem that this continuous subharmonic function $g(z)$ on $\mathbb{C}$ is actually Hölder continuous.

When C_P is contained in K_P, then any Böttcher function $\phi(z)$ can be extended to a conformal map of I_P onto $\{|z| > 1\}$. In general, set

$$M = \max\{g(z_0) \mid z_0 \in C_P\};$$

then we can extend $\phi(z)$ to a conformal map of $\{z \in \mathbb{C} \mid g(z) > M\}$.

Indeed, if $r > M$, then $\{g(z) = r\}$ is a simple closed curve, and hence we can find a (multi-valued) conjugate harmonic function $g^*(z)$ of $g(z)$ such that $G(z) = \exp(g(z) + ig^*(z))$ is single valued on $\{|z| > r\}$. Choosing the additive constant of $g^*(z)$ so that $G(z) = \phi(z)$ near ∞, we conclude with the assertion. In particular, if $C_P \subset K_P$, Proposition 1.2.2 implies that $G(z)$ is a conformal map of I_P onto $\{|z| > 1\}$.

Now we turn to the case of a finite fixed point, which we assume to be 0 in the rest of this section. We can generalize Böttcher's theorem as follows.

Theorem 1.2.3 *Let $f(z)$ be a holomorphic function in a neighborhood of the origin with the Taylor expansion*

$$f(z) = c_0 z^k + c_1 z^{k+1} + \cdots \quad (c_0 \neq 0).$$

Then there exist a neighborhood U of the origin and a conformal map $\phi : U \to \mathbb{C}$ fixing the origin and satisfying

$$\phi(f(z)) = \{\phi(z)\}^k.$$

Moreover, such a ϕ is uniquely determined up to multiplication by a $(k-1)$th root of 1.

Again, we call such a $\phi(z)$ a *Böttcher function* for f (at 0). The proof is the same as that of Theorem 1.2.1 and hence omitted.

Remark We can derive Theorem 1.2.1 from Theorem 1.2.3 by taking the conjugate by $T(z) = 1/z$.

Note that Theorem 1.2.3 treats the case of a fixed point in C_P for a monic polynomial P. When a given fixed point does not belong to C_P, or more generally, when a holomorphic function $f(z)$ in a neighborhood of the origin fixes the origin and satisfies $f'(0) \neq 0$, we have the Taylor expansion

$$f(z) = \lambda z + c_2 z^2 + \cdots \quad (\lambda \neq 0).$$

We call this λ the *multiplier* of f at the fixed point 0. The case that $\lambda = 0$ has been treated in Theorem 1.2.3.

Theorem 1.2.4 (Koenigs) *Suppose that a function $f(z)$ holomorphic near the origin has the Taylor expansion*

$$f(z) = \lambda z + c_2 z^2 + \cdots \quad (0 < |\lambda| < 1).$$

Then there are a neighborhood U of the origin and a conformal map $\phi(z)$ of U such that $\phi(0) = 0$ and that

$$\phi \circ f(z) = \lambda \phi(z) \quad (z \in U). \tag{1.3}$$

Here, $\phi(z)$ is unique up to multiplicative constants.

Definition We call the equation (1.3) the *Schröder equation* for f. When the Schröder equation has a solution, then we say that f is *linearizable* at 0.

Proof of theorem Set

$$\phi_n(z) = \lambda^{-n} f^n(z)$$

for every n and we have

$$\phi_n \circ f(z) = \lambda^{-n} f^{n+1}(z) = \lambda \phi_{n+1}(z).$$

Fix an η such that $\eta^2 < |\lambda| < \eta < 1$, and we can choose a positive δ so small that

$$|f(z)| < \eta |z| \quad \text{and} \quad |f(z) - \lambda z| \leq c|z|^2$$

on $\{|z| < \delta\}$, where $c = |c_2| + 1$. Further, set $\rho = \eta^2/|\lambda|$ (< 1), and we obtain that

$$\begin{aligned} |\phi_{n+1}(z) - \phi_n(z)| &= |\lambda^{-n-1}\{f(f^n(z)) - \lambda f^n(z)\}| \leq |\lambda|^{-n-1} c |f^n(z)|^2 \\ &\leq c\rho^n |\lambda|^{-1} |z|^2. \end{aligned}$$

Thus $\phi_n(z)$ converges to

$$\phi(z) = \phi_1(z) + \sum_{n=1}^{\infty} \{\phi_{n+1}(z) - \phi_n(z)\}$$

uniformly on $\{|z| < \delta\}$, and it is clear that $\phi(z)$ satisfies the Schröder equation. The uniqueness follows by comparing the coefficients. □

Remark When $|\lambda| > 1$, we can show the same assertion by considering the inverse function.

1.2.2 Schröder equation for λ *with* $|\lambda| = 1$

In the case that $|\lambda| = 1$, the situation is very complicated. We gather several basic facts without proofs, which will be found in the books referred to before.

First, for almost every such λ, the corresponding Schröder equation has a solution.

Theorem 1.2.5 (Siegel) *For almost every* λ *on the unit circle* $\{|z| = 1\}$*, every function* $f(z)$ *represented by a convergent power series*

$$f(z) = \lambda z + c_2 z^2 + \cdots$$

is linearizable at 0*, i.e., there exist a neighborhood* U *of the origin and a conformal map* $\phi(z)$ *such that*

$$\phi(f(z)) = \lambda \phi(z) \quad (z \in U).$$

Remark Actually, Siegel showed that, if $t \in \mathbb{R} - \mathbb{Q}$ and is *Diophantine*, i.e., there are constants $c > 0$ and $b < +\infty$ such that

$$\left| t - \frac{p}{q} \right| \geq \frac{c}{q^b}$$

for every integer p and positive integer q, then $f(z)$ with $\lambda = e^{2\pi i t}$ is linearizable at 0.

Here, if we take $b > 2$, then the total length of the set of non-Diophantine numbers in $[0, 1]$ is not greater than $\sum_{q=N}^{\infty} 2cq^{1-b}$, which is the total length of intervals $\{t \in [0, 1] \,|\, |t - (p/q)| < cq^{-b}\}$ with $q \geq N$ for every N, and which tends to 0 as $N \to +\infty$. Hence almost every number is Diophantine.

On the other hand, at every *rationally indifferent* fixed point, that is every fixed point with a multiplier $\lambda = e^{2\pi i t}$ with $t \in \mathbb{Q}$, the function is not linearizable at 0. In particular, λs corresponding to the non-linearizable case are dense in the unit circle.

Theorem 1.2.6 (Petal theorem of Leau and Fatou) *For every function $f(z)$ represented by a convergent power series*

$$f(z) = z + c_{p+1} z^{p+1} + \cdots \quad (c_{p+1} \neq 0)$$

at 0*, there are* $2p$ *domains* $\{U_j, V_j\}_{j=1}^{p}$ *such that*

(i) *for* $U = \bigcup_{j=1}^{p} U_j$,

$$f(\overline{U}) \subset U \cup \{0\}, \quad \bigcap_{n=1}^{\infty} f^n(\overline{U}) = \{0\},$$

and for $V = \bigcup_{k=1}^{p} V_k$, $(f|_V)^{-1}$ *is univalent on* V*, and*

$$(f|_V)^{-1}(\overline{V}) \subset V \cup \{0\}, \quad \bigcap_{n=1}^{\infty} (f|_V)^{-n}(\overline{V}) = \{0\},$$

(ii) *domains* $\{U_j\}_{j=1}^{p}$ *are pairwise disjoint and so are* $\{V_j\}_{j=1}^{p}$*, and*
(iii) *the domain* $\{0\} \cup U \cup V$ *is a neighborhood of* 0.

Definition We call a component of U satisfying condition (i) an *attracting petal* of $f(z)$ at the origin. Also, an attracting petal of $(f|_V)^{-1}(z)$ such as V_k in Theorem 1.2.6 is called a *repelling petal* of f.

Remark When f has the form

$$f(z) = \lambda z + c_{p+1} z^{p+1} + \cdots \quad (c_{p+1} \neq 0)$$

with a primitive mth root of unity λ, consider $f^m(z)$, and we have a similar assertion as above. In particular, $f(z)$ has a family of attracting petals, divides it into invariant subfamilies and permutes the elements of each subfamily cyclically.

Thus, when $\lambda = e^{2\pi i t}$ with a rational t, the Schröder equation for f at the origin has no solution.

Example 1.2.1 *Let*

$$f(z) = z + c_2 z^2 + \cdots \qquad (c_2 \neq 0)$$

and define $T(z) = -1/(c_2 z)$*. Further we set*

$$\begin{aligned} \Omega_+ &= \{|\mathrm{Im}\, z| > 2c - \mathrm{Re}\, z\}, \\ \Omega_- &= \{|\mathrm{Im}\, z| > 2c + \mathrm{Re}\, z\}, \end{aligned}$$

with a sufficiently large c*. Then* $T(\Omega_+)$ *is an attracting petal of* $f(z)$ *at the origin and* $T(\Omega_-)$ *is a repelling petal of* $f(z)$ *at the origin.*

Verification By taking a conjugate by $T(z)$, we have

$$g(z) = T \circ f \circ T^{-1}(z) = z + 1 + \frac{a_1}{z} + \cdots.$$

We choose c so that

$$|g(z) - z - 1| < 1/\sqrt{2}$$

if $|z| \geq c$. Then $g(z)$ is nearly a translation near ∞. Hence a simple computation shows that $\overline{g(\Omega_+)} \subset \Omega_+ \cup \{\infty\}$, and $T(\Omega_+)$ is an attracting petal of $f(z)$ at the origin. Similarly, we see that $T(\Omega_-)$ is a repelling petal of $f(z)$ at the origin.

As examples of quadratic polynomials $z^2 + c$ having attracting petals, we show as Figure 1.5 the cases that (a) $c = 1/4$ and (b) $c = 0.31\cdots + i0.03\cdots$.

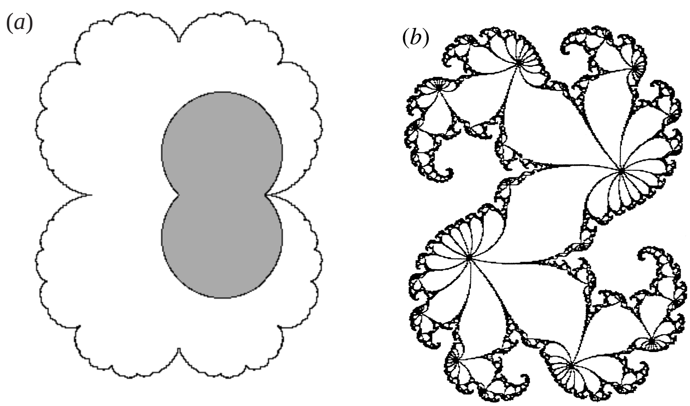

Fig. 1.5. Julia sets with parabolic basins

Remark The explanation in the previous example indicates that, for any attracting petal U of $f(z)$, there exists a conformal map $\phi(z)$ of U which satisfies the *Abel equation*

$$\phi(f(z)) = \phi(z) + 1 \quad (z \in U).$$

We call such a $\phi(z)$ a *Fatou function* for the attracting petal U.

As another typical case where the Schröder equation has no solutions, we cite the following.

Theorem 1.2.7 (Cremer) *For every λ with $|\lambda| = 1$, set*

$$Q_\lambda(z) = \lambda z + z^2.$$

Then the set of λ such that the Schröder equation for Q_λ has no solutions is a generic set, that is one which can be represented as the intersection of at most countably many open dense subsets, in $\{|\lambda| = 1\}$.

Definition For a function $f(z)$ holomorphic near z_0 which has the Taylor expansion

$$f(z) = z_0 + \lambda(z - z_0) + c_2(z - z_0)^2 + \cdots \quad (\lambda = e^{2\pi i t};\ t \in \mathbb{R} - \mathbb{Q}),$$

z_0 is called an *irrationally indifferent* fixed point. Further, we say that z_0 is a *Cremer point* of $f(z)$ if the Schröder equation for $f(z)$ at z_0 has no solutions.

Corollary *The set of λ such that Q_λ has a Cremer point is dense in the unit circle.*

Remark Suppose that 0 is a fixed point of a polynomial P with the multiplier λ with $|\lambda| = 1$. Then it is easy to see that, if P is linearizable at 0, then 0 does not belong to J_P. The converse is also true. See Theorem 2.1.9.

Finally, we note the relation between Cremer points and the continued fractional expansion in the case of quadratic polynomials.

Definition Let $t \in \mathbb{R} - \mathbb{Q}$, and consider the continued fractional expansion

$$t = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}},$$

which induces the sequence $\{q_n\}_{n=1}^{\infty}$ of the denominators of the rational approximation of t, i.e. $\{q_n\}$ are determined by the following equations inductively:

$$q_0 = 1, q_1 = a_1, \ldots, q_{n+1} = q_{n-1} + q_n a_{n+1}, \ldots.$$

We say that t is a *Brjuno number* if

$$\sum_{n=1}^{\infty} \frac{1}{q_n} \log q_{n+1} < \infty.$$

Theorem 1.2.8 (Brjuno–Yoccoz) *Let $\lambda = e^{2\pi i t}$ with $t \in \mathbb{R} - \mathbb{Q}$, and $Q_\lambda(z) = \lambda z + z^2$. Then the Schröder equation for Q_λ has a solution near the origin if and only if t is a Brjuno number.*

In general, t is a Brjuno number if and only if every function $f(z)$ represented by a convergent power series

$$f(z) = \lambda z + c_2 z^2 + \cdots$$

is linearizable at 0. For the details, see Yoccoz (1996).

1.3 Quadratic polynomials and the Mandelbrot set

In the case of quadratic polynomials $P_c(z) = z^2 + c$, Theorem 1.1.4 means that K_{P_c} is connected if and only if the orbit of 0 is bounded. So we define

$$\mathcal{M} = \{c \in \mathbb{C} \mid \{P_c^n(0)\}_{n=1}^{\infty} \text{ is a bounded sequence}\},$$

and call it the *Mandelbrot set*. See Figure 1.6.

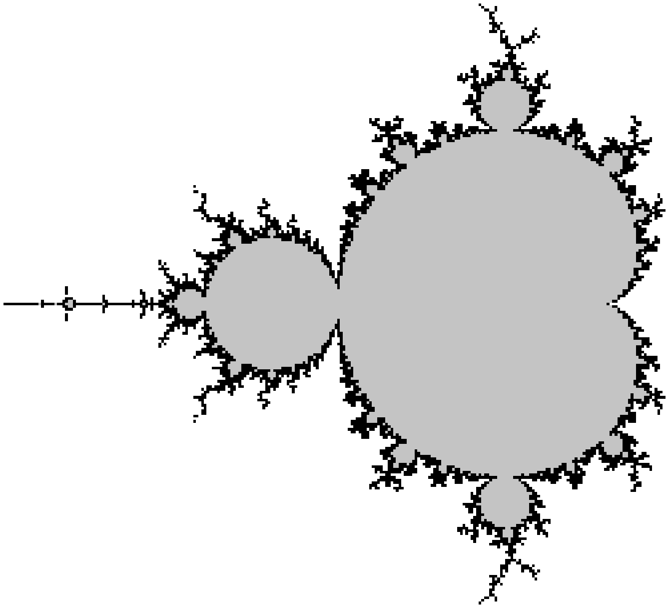

Fig. 1.6. Mandelbrot set