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Holomorphic Dynamics

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521662581

Originally published in Japanese as Fukuso-rikigakukei josetsu
by S. Morosawa, M. Taniguchi and T. Ueda by Baifukan, Tokyo

This revised English edition by S. Morosawa, Y. Nishimura, M. Taniguchi, T. Ueda
First published by Cambridge University Press

First published 2000

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication Data

Holomorphic dynamics / S. Morosawa, Y. Nishimura, M. Taniguchi, T. Ueda.
p. cm.
Includes bibliographic references.
ISBN 0 521 66258 3
1. Differentiable dynamical systems. 2. Holomorphic mappings.
3. Functions of complex variables. I. Morosawa, S. (Shunsuke),
1958– .
QA614.8.H66 1999
512'.352–dc21 99-14417 CIP

ISBN 978-0-521-66258-1 Hardback

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Preface

Recently, various phenomena caused by complex dynamics have drawn much attention not only from mathematicians but also from scientists over wide areas. Here we mean by complex dynamics vaguely the actions induced by iteration of analytic maps of complex number spaces. The researches on them are developing rapidly and relevant papers have become enormous in number. Nowadays, there are several excellent introductions, especially to iteration of rational functions, which is one of the central topics of complex dynamics, such as those written by Milnor, Beardon, Carleson and Gamelin, Steinmetz, and McMullen.

This book is a more comprehensive introduction to holomorphic dynamics: complex dynamics induced by holomorphic maps, including the cases of entire functions, Kleinian groups and polynomial functions of several complex variables such as complex Hénon maps, as well as the case of rational functions. Also, we emphasize the substantial value of the classical theory of complex analysis in research in holomorphic dynamics.

First, in Chapter 1, we start with several typical fractal sets appearing under iteration of polynomials, and then summarize fundamental facts on the behavior of polynomials near fixed points under iteration. We also explain the Mandelbrot set for quadratic polynomials and some results concerning Hausdorff measures and the logarithmic capacity. Finally, we introduce the concept of polynomial-like maps, one of the new fundamental concepts in this field.

In Chapter 2, we give a dichotomy of the Riemann sphere or the complex plane, respectively, with respect to dynamics of a given rational or entire function, from the viewpoint of the normal family. On one part called the Fatou set, the action is tame, but it is chaotic on the other called the Julia set. After summing up classical results on value distribution theory, we explain some fundamental properties of these sets valid for both cases, of rational functions and of entire functions, including denseness of repelling periodic points in Julia sets, and a general version of the non-wandering domains theorem, after introducing the Teichmüller spaces of holomorphic dynamics.

In Chapter 3, we explain the dynamics of entire functions, considered as holomorphic maps of the complex plane into itself, and give several examples of wandering domains and so-called Baker domains, which never appear in the case of rational functions. Also we discuss fundamental features of the exponential family and other tame families of entire functions, including the Cantor-bouquet structure and topological completeness.

Chapter 4 starts with a survey of recent topics about the classical Newton method. Next, we give a self-contained exposition (fairly independent of Chapter 2) of the standard theory due to Fatou, Julia and Sullivan on iteration of rational functions. Also we include a survey of Shishikura's fundamental theorems, and discuss closely so-called hyperbolic rational functions.

Now, we may consider the group action of Kleinian groups (discrete subgroups of the Lie group consisting of linear fractional transformations) as dynamical systems, which are nowadays called conformal dynamics. In this case, the Julia sets and the Fatou sets are conventionally called the limit sets and the ordinary sets, respectively. But it is classically recognized that properties of these sets resemble each other. Having gathered up such analogies, we now obtain a fairly long list, which is called Sullivan's dictionary, some of which we discuss in Chapter 5.

Thus Chapters 1–5 are devoted to the case of holomorphic dynamics on domains in the Riemann sphere. The theory in this case has a long history and is rather mature, as is seen from the fact that there are many excellent books on it. Hence we intend also to include a modern survey of recent progress in these chapters.

On the other hand, Chapters 6–9 treat the case of two complex variables, which is still a frontier in research on holomorphic dynamics. As a consequence, there are few guide books which give comprehensive introductions to such topics. Thus we intend to give a self-contained exposition of holomorphic dynamics of two variables. Thus, these chapters can be read almost independently of the preceding ones, though many motivations and many ways of thinking are in the same flavor.

Now, Chapter 6 starts with a survey of fundamental facts of the complex analysis of several variables. Next, after explaining the behavior near fixed points of a holomorphic automorphism of the multi-dimensional complex number space, we construct several examples of Fatou–Bieberbach domains, namely, proper domains biholomorphic to the whole space. Finally we introduce stable and unstable submanifolds near saddle (fixed) points as a local theory.

In Chapter 7, we first classify polynomial automorphisms of two variables and show that complex Hénon maps are of essential importance. Next, we concentrate on complex Hénon maps in the remainder of this chapter. We divide the whole space into the tame part consisting of escaping points and the chaotic one consisting of non-escaping points as in Chapter 1 and explain structures of each part. In particular, we represent the chaotic part by using the theory of symbolic dynamics.

Chapter 8 is an introduction devoted to the pluripotential theory, which is crucial for the measure theoretic treatment of the dynamics of several complex variables. First, we introduce the notion of currents and some kind of Lelong theorem on the solution of the $\partial\bar{\partial}$-problem. We discuss fundamental properties of the Green functions and the stable and the unstable currents, and after proving the fundamental convergence theorem we explain some of the pluripotential theoretic properties of polynomial automorphisms.

Finally, in Chapter 9, we explain, as the interaction between the pluripotential theory and the dynamics of polynomial automorphisms, fundamental global properties of important invariant sets, such as the basins of Siegel disks or Herman rings, when the absolute value of the Jacobian is less than 1. Finally, we discuss hyperbolic polynomial automorphisms.

Chapters 1–5 are written by Shunsuke Morosawa and Masahiko Taniguchi, Chapters 6 and 7 are by Tetsuo Ueda, and Chapters 8 and 9 are by Yasuichiro Nishimura. Chapters 1–7 are a revised, updated, and extended translation of a book written by Morosawa, Taniguchi and Ueda in Japanese, published by Baifukan. We express heartfelt gratitude to Shigehiro Ushiki and Yasuhiro Gotou for allowing us to use computer graphics which they produced, and Mitsuhiro Shishikura for his valuable comments.

February, 1998
The authors