1

Dynamics of polynomial maps

A simple mechanism can make a chaos. The case of iteration of polynomial maps is no exception. Indeed an extremely simple map, $z \to z^2 + c$, can create deep confusion. Understanding of, or, more modestly, an attempt to understand, such phenomena is the theme of this book.

More precisely, consider a monic polynomial

$$P(z) = z^k + a_1 z^{k-1} + \cdots + a_{k-1} z + a_k$$

of degree k. Then P can be considered as a self-map of $\mathbb{C}$, and the issue is to discuss what happens when we apply P to a given point z iteratively. Set

$$z_0 = z, z_1 = P(z), \ldots, z_n = P(z_{n-1}), \ldots,$$

and consider the behavior of the sequence $\{z_n\}_{n=1}^{\infty}$ or the orbit of z. Here, one might take some stable dependence on the initial value z for granted. But this is not true in general, and the set of unstable initial values is, almost always, incredibly complicated, and nowadays is called a fractal. Moreover, for almost every such initial value, the behavior is chaotic enough.

First, in §1.1, we give several typical examples of fractal sets caused by polynomial maps, and explain basic properties of them. Then §1.2 summarizes local theory near a fixed point of a general holomorphic map.

Turning to quadratic polynomial maps as above, we introduce the Mandelbrot set, which in itself is a fractal and represents the very interesting family of quadratic polynomial maps, in §1.3.

As the fundamental tools to measure how complicated the set considered is, we introduce in §1.4 the Hausdorff measures and the logarithmic capacity, and give some basic facts about them.

Finally, in §1.5, we define a polynomial-like map, which is one of the fundamental concepts in the modern theory of complex dynamics, and explain some basic results.

In this chapter, we explain not only basic facts on the dynamics of polynomials, but also several central issues about complex dynamics. In particular, we include many important results without proofs. As general references for those results, we cite books by Beardon (1991), Carleson and Gamelin (1993), Milnor (1990), and Steinmetz (1993).

1.1 The set of escaping points for a polynomial

1.1.1 Typical examples of polynomials

Fix a monic polynomial

$$P(z) = z^k + a_1 z^{k-1} + \cdots + a_{k-1} z + a_k$$

of degree $k \geq 2$, and consider the sequence of $\{P^n\}_{n=0}^{\infty}$, where and in what follows, we set

$$P^0(z) = z,\ P^1(z) = P(z), \ldots,\ P^n(z) = P(P^{n-1}(z)).$$

Then for every z with sufficiently large $|z|$, we have

$$|z| < |P(z)| < |P^2(z)| < \cdots.$$

In fact, choose an $R > 2$ so large that $|P(z)| > |z|^k/2$ for every z with $|z| > R$. Then inductively, we see that

$$R < |z| < |z|^k/2 < |P(z)| < \cdots < |P^{n-1}(z)| < |P^{n-1}(z)|^k/2 < |P^n(z)|.$$

Thus for every such z, $P^n(z)$ tends to ∞ in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition We set

$$I_P = \left\{ z \in \mathbb{C} \,\middle|\, \lim_{n\to\infty} P^n(z) = \infty \right\},$$

and call it the *set of escaping points* for P. Also we set

$$K_P = \mathbb{C} - I_P$$

and call it the *filled Julia set* of P. The boundary $J_P = \partial K_P$ of K_P is called the *Julia set* of P.

Remark Since $k \geq 2$, any polynomial

$$P(z) = a_0 z^k + a_1 z^{k-1} + \cdots + a_{k-1} z + a_k \quad (a_0 \neq 0)$$

is affine conjugate to a monic polynomial of degree k. More precisely, set $A(z) = \omega z$, where ω is a $(k-1)$th root of a_0, and we see that $A \circ P \circ A^{-1}(z)$ is a monic polynomial of degree k.

The following example is clear.

Example 1.1.1 *Let $P(z) = z^2$. Then we have*

$$I_P = \{|z| > 1\}, \quad K_P = \{|z| \leq 1\}, \quad J_P = \{|z| = 1\}.$$

Example 1.1.2 *Let $P(z) = z^2 - 2$. Then we have*

$$I_P = \mathbb{C} - [-2, 2], \quad K_P = J_P = [-2, 2],$$

where $[-2, 2]$ *denotes the interval* $\{x \in \mathbb{R} \mid -2 \leq x \leq 2\}$.

Verification Set $B(z) = z + 1/z$, and we can take the single valued branch of B^{-1} from $\mathbb{C} - [-2, 2]$ onto $\{|z| > 1\}$. A simple calculation shows that $P(z) = B \circ P_0 \circ B^{-1}(z)$ with $P_0(z) = z^2$. Hence by Example 1.1.1, we see that I_P contains $\mathbb{C} - [-2, 2]$. On the other hand, it is clear that $P([-2, 2]) = [-2, 2]$, and hence the assertions follow.

Recall that P restricted to $[-2, 2]$ is a typical unimodal map which represents an action that expands the interval to twice its length and then folds it into two.

In general, J_P has a very complicated shape. Actually, the above examples are exceptions. We show the variety of the Julia sets by examples.

Remark The reasons why the Julia sets in the examples below have such shape are not easy, and need several results discussed later. But for the sake of convenience, we include brief explanations.

We start by clarifying what we consider as fractal sets. There are several ways to define fractal sets. Self-similarity relates closely to fractal sets. But there exist so many complicated sets, for which self-similarity is not so clear or seems to be absent. From a quantitative viewpoint, we may call every simple curve with Hausdorff dimension, defined in §1.4, greater than 1 a fractal curve. But we take a looser definition (also see §4.1).

Definition Let E be a closed subset of $\mathbb{C}$ without interior. We say that E is a *fractal set* if E cannot be represented as a countable union of rectifiable curves.

We say that a domain D in $\widehat{\mathbb{C}}$ is a *fractal domain* if the boundary ∂D is a fractal set.

As to fractal sets in the multi-dimensional complex number space, we will discuss these in Chapters 6–9.

Even in the case that the Julia set J_P is a simple closed curve, J_P is terribly complicated except for trivial cases.

To give such examples, we perturb $P(z) = z^2$ in Example 1.1.1. That is, consider $P_c(z) = z^2 + c$ with sufficiently small c. Then simple computations show that there is a fixed point z_c, a solution of $P(z) = z$, near $z = 0$ and hence the value of the derivative at z_c is also near to 0. Moreover, we will show that the structure of J_{P_c} is circle-like.

Example 1.1.3 *When $|c|$ is sufficiently small, the Julia set J_{P_c} of $P_c(z) = z^2+c$ is a simple closed curve, but has a very complicated shape, called a fractal circle. See Figure 1.1(a), the Julia set of $z^2 + c$, $c = 0.59\cdots + i0.43\cdots$, and Figure 1.1(b), that of $z^2 + c$, $c = 0.33\cdots + i0.07\cdots$.*

Verification By the corollary to Theorem 1.5.1, each Julia set is a quasicircle. In particular, it is a simple closed curve. Theorem 1.4.7 implies that it is a fractal set.

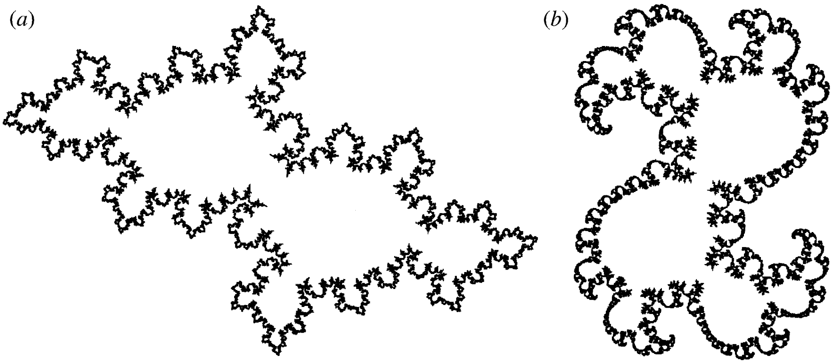

Fig. 1.1. Jordan curves as Julia sets

Definition We say that a compact set K in $\widehat{\mathbb{C}}$ is *locally connected* if, for every $z_0 \in K$ and every neighborhood U of z_0, $U \cap K$ contains a connected neighborhood of z_0 in K.

A connected and locally connected compact set is arcwise connected. For some conditions on local connectedness, see §4.4.

Example 1.1.4 *Set*

$$E = \left\{ x + iy \,\middle|\, 0 < x < 1,\, y = \sin\frac{1}{x} \right\},$$

and the closure of E is not locally connected.

Definition We call a compact set K a *dendrite* if K is a connected and locally connected compact set without interior whose complement $\mathbb{C}-K$ is connected.

Example 1.1.5 *A typical example of a dendrite is the (filled) Julia set of z^2+i. As other examples, we cite in Figure 1.2 the (filled) Julia sets of*

(*a*) $z^2 + c, \quad c \in \mathbb{R},\ c^3 + 2c^2 + 2c + 2 = 0,$
(*b*) $z^3 + c, \quad c = \sqrt{\omega - 1},\ \omega^2 + \omega + 1 = 0.$

Verification These Julia sets are connected by Theorem 1.1.4, for the orbit of the unique critical value 0 is preperiodic and hence bounded. They have no interior points, which can be shown as in the proof of Theorem 4.2.18. Further, since they are subhyperbolic, defined in §4.4, the Julia sets are locally connected.

When the Julia set is a dendrite, it coincides with the filled Julia set. There is another typical case that these are coincident with each other. It is the case that the Julia set of a polynomial is a Cantor set. We will define and discuss Cantor sets in §1.4. Here we temporarily call a set E a Cantor set if E has no isolated points, and every connected component of E consists of a single point. Then it is easy to see that such an E has uncountably many connected components.

By definition, if a closed set without interior has uncountably many connected components, then it is a fractal set. We give in Figure 1.3 the Julia sets of

(*a*) $z^2 + c, \quad c = 0.38\cdots + i0.22\cdots,$
(*b*) $z^2 + c, \quad c = -0.82\cdots + i0.33\cdots.$

These Julia sets are Cantor sets by Theorem 1.1.6, for the c are outside of the Mandelbrot set defined in §1.3.

In the case of polynomials P of degree greater than 2, the filled Julia set may have interior points even if it has uncountably many connected components. We give the following example.

Example 1.1.6 *The Julia set of $P(z) = z^3 + az$ with $a = 0.038 + i1.95$ consists of uncountably many components and countably many components have interior points. See Figure 1.4.*

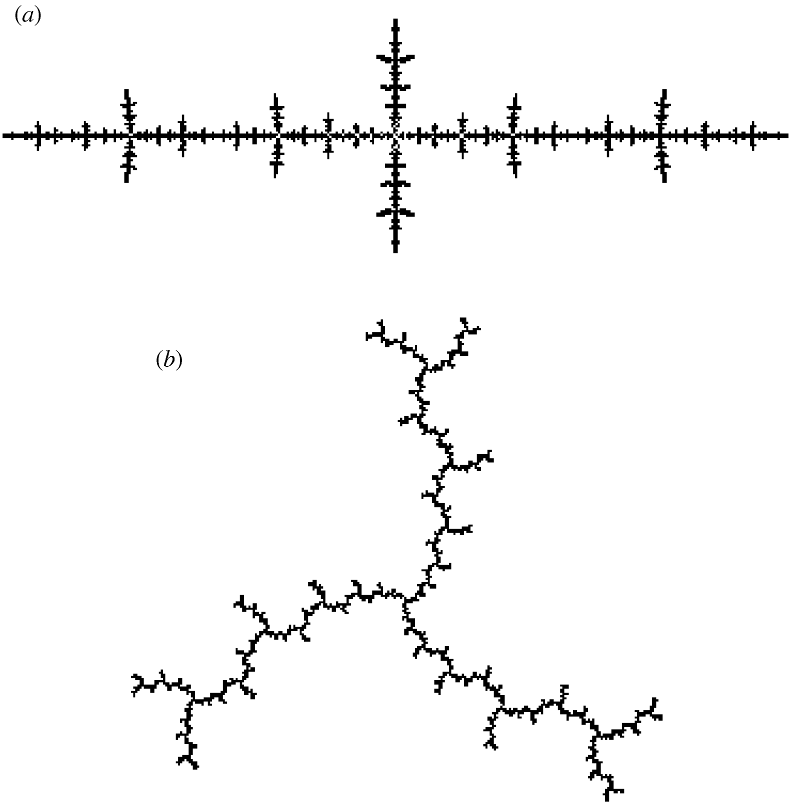

Fig. 1.2. Dendrites as Julia sets

Verification Since Theorem 2.3.6 shows that Julia sets are perfect sets, a disconnected Julia set consists of uncountably many components. To see the filled Julia set of P has countably many connected components which have interior points, we need only pursue the orbit of the finite critical points $\pm\sqrt{-a/3}$, as is seen from the fact proved in Chapter 2. We remark that the shape of each component depends on the polynomial induced by a suitable renormalization, which is defined in §1.5.

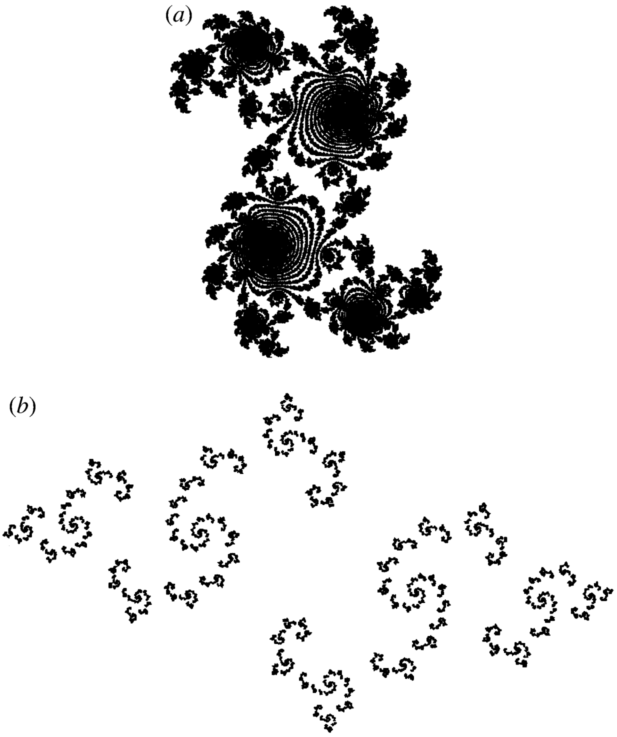

Fig. 1.3. Cantor sets as Julia sets

1.1.2 Basic properties of the filled Julia set

We have seen that the Julia set may have incredibly varied shapes even restricted to simple polynomial maps. One purpose of this book is to discuss such variety of shapes and characters of the Julia sets, though we stress explanation of the unified nature which a fairly large class of complex dynamics are equipped with.

We start with the following elementary fact.

Theorem 1.1.1 *Let $P(z)$ be a polynomial of degree not less than 2. Then the filled Julia set K_P is a non-empty compact set.*

Moreover, I_P is a domain, that is a connected open set.

Remark When $P(z) = z + c$ $(c \neq 0)$, it is clear that K_P is empty. The case of a general linear map is left to the reader.

Fig. 1.4. Disconnected Julia set with non-empty interior

Proof of theorem Since the degree of P is not less than 2, P always has a fixed point, which belongs to K_P by definition. Hence K_P is non-empty.

Next, as noted before, setting $V = \left\{z \in \mathbb{C} \,\middle|\, |z| > R\right\}$ for a sufficiently large R, we have $P(V) \subset V$ and $V \subset I_P$. On the other hand, for every $z_0 \in I_P$, the orbit elements $P^n(z_0)$ tend to ∞, and hence there is an N such that $P^N(z_0) \in V$. Thus we conclude that

$$I_P = \bigcup_{n=1}^{\infty} (P^n)^{-1}(V), \tag{1.1}$$

which in particular implies that I_P is an open set and that K_P is a bounded closed set.

Finally, if there exists a connected component of $(P^n)^{-1}(V)$ disjoint from V, then it should be bounded, for $(P^n)^{-1}(V)$ contains V. This contradicts the maximal principle, and hence we conclude that $(P^n)^{-1}(V)$ is a domain containing V. Thus the above equation shows the assertion. □

The equation (1.1) also implies the following.

Theorem 1.1.2 (Complete invariance) *Let $P(z)$ be a polynomial of degree not less than 2. Then the sets I_P, K_P and J_P are completely invariant under the action by P, i.e., letting E be one of these sets, we have*

$$P(E) = P^{-1}(E) = E.$$

Proof By (1.1), we see that $P^{-1}(I_P) = I_P$. Since $P((P^n)^{-1}(V)) = (P^{n-1})^{-1}(V)$, we also have that $P(I_P) = I_P$. Thus I_P is completely invariant, and so is the complement K_P. Since P is a continuous open map, J_P is also completely invariant. □

Theorem 1.1.2 implies that every point of K_P has a bounded orbit. Hence we can characterize K_P as the set of points with bounded orbits.

Definition A closed set E in $\widehat{\mathbb{C}}$ is called *perfect* if E has no isolated points.

A domain D in $\widehat{\mathbb{C}}$ is *simply connected* if the complement $\widehat{\mathbb{C}} - D$ is connected.

Theorem 1.1.3 *Let $P(z)$ be a polynomial of degree not less than 2. Then the filled Julia set K_P is a perfect set.*

Moreover, every connected component of the interior of K_P is simply connected.

Proof Suppose that there is an isolated point z_0 of K_P. Then we can draw a simple closed curve C in I_P such that the intersection of the interior W of C in $\mathbb{C}$ and K_P consists only of the point z_0.

Now take such a V as in the proof of Theorem 1.1.1. Then by (1.1), we can find N such that $P^N(C) \subset V$. Since $P^N(z_0)$ is not contained in V, $P^N(W)$ should contain $\mathbb{C} - V$. Then complete invariance of K_P means that $K_P = \{z_0\}$. In particular, $P^{-1}(z_0) = \{z_0\}$, and hence $P(z)$ should have the form $a(z - z_0)^k + z_0$. But then z_0 should belong to the interior of K_P, which is a contradiction.

Next, if there were a connected component of the interior of K_P which were not simply connected, then $I_P \cup J_P = \overline{I_P}$ would not be connected, which contradicts Theorem 1.1.1. □

Remark For every $z \in I_P$, the closure of $\bigcup_{n=1}^{\infty}(P^n)^{-1}(z)$ contains J_P, and for every $z \in J_P$, we have

$$\overline{\bigcup_{n=1}^{\infty}(P^n)^{-1}(z)} = J_P.$$

These facts follows from the non-normality of $\{P^n\}$ on the Julia set (see Theorem 2.3.4), and give one of the standard ways to draw the Julia set.

Definition The solutions of $P'(z) = 0$ are called *critical points* of $P(z)$. We denote by C_P the set of all critical points of $P(z)$, and call it the *critical set* of

$P(z)$:

$$C_P = \{z \in \mathbb{C} \mid P'(z) = 0\}.$$

Now whether $I_P \cup \{\infty\}$ is simply connected or not depends on the location of the critical set.

Theorem 1.1.4 *Let $P(z)$ be a polynomial of degree not less than 2. Then the filled Julia set K_P is connected if and only if $I_P \cap C_P = \emptyset$.*

Proof As in the proof of Theorem 1.1.1, fix R so that $V = \{|z| > R\}$ is contained in I_P.

First, suppose that $I_P \cap C_P$ is the empty set. Then P^n gives a smooth k^n-sheeted covering of $V_n = (P^n)^{-1}(V)$ onto V, where k is the degree of P. Since $V \cup \{\infty\}$ is simply connected, so is every $V_n \cup \{\infty\}$. Hence the union $I_P \cup \{\infty\}$ of increasing domains $V_n \cup \{\infty\}$ is simply connected.

Next suppose that $I_P \cap C_P$ is not empty, and let N be the minimum of n such that $V_n \cap C_P \neq \emptyset$. Then applying Lemma 1.1.5 below to the proper holomorphic map $P : (V_N - \overline{V_{N-1}}) \to (V_{N-1} - \overline{V_{N-2}})$, we see that V_N is not simply connected. Hence K_P is not connected. □

Lemma 1.1.5 (Riemann–Hurwitz formula for domains) *Let D_1 and D_2 be domains in $\widehat{\mathbb{C}}$ whose boundaries consist of a finite number of simple closed curves. Let $f(z)$ be a proper holomorphic map of D_1 onto D_2. Then:*

(i) *Every $z \in D_2$ has the same number k of preimages including multiplicity.*
(ii) *Denote by N the number of critical points of f in D_1 including multiplicity. Then*

$$(2 - d_1) = k(2 - d_2) - N,$$

where d_j is the number the boundary components of D_j.

In particular, when both the D_j are simply connected, f has at most $k - 1$ critical points.

The number k in the lemma is called the degree of f.

Proof The first assertion follows since f is a proper open map. The second assertion follows by applying Euler's formula to suitable triangulations of D_j. □

A typical example of disconnected K_P is those such as in Figure 1.3. We say that a compact set E is *totally disconnected* if every connected component of E consists of a single point.