The projective, Möbius, Laguerre, and Minkowski planes over the real numbers are just a few examples of a host of fundamental classical topo logical geometries on surfaces that satisfy an axiom of joining. This book summarises all known major results and open problems related to these classical geometries and their close (non-classical) relatives.

Topics covered include: classical geometries; methods for constructing non-classical geometries; classifications and characterisations of geometries. This work is related to a host of other fields including interpolation theory, convexity, differential geometry, topology, the theory of Lie groups and many more. The authors detail these connections, some of which are well-known, but many much less so.

Acting both as a referee for experts and as an accessible introduction for beginners, this book will interest anyone wishing to know more about incidence geometries and the way they interact.

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

FOUNDING EDITOR G.-C. ROTA

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge, CB2 2RU, UK
40 West 20th Street, New York, NY 10011–4211, USA
10 Stamford Road, Oakleigh, VIC 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2001

Printed in the United Kingdom at the University Press, Cambridge

Typeface Computer Modern 10/13pt *System* LaTeX 2$_\varepsilon$ [UPH]

A catalogue record of this book is available from the British Library

ISBN 0 521 66058 0 hardback

Cambridge University Press
978-0-521-66058-7 - Geometries on Surfaces
Burkard Polster and Gunter Steinke
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To Anu and Marina

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Preface

What This Book Is All About

'Geometries on surfaces'—what do you think of when you read such a title? Whatever it is will depend to a large extent on your background in mathematics. Our background is in incidence geometry, and, even if we were not the authors of this book, we would first think of examples such as the Euclidean plane and the geometry of circles on a sphere. These two geometries have a number of features in common. For example, the point sets of both geometries are surfaces, the lines or circles are curves that are nicely embedded in these surfaces, and both geometries satisfy an 'axiom of joining'—in the Euclidean plane two points are contained in exactly one line and in the geometry on the sphere three points are contained in exactly one circle.

The Euclidean plane and the geometry of circles on a sphere are just two examples of a host of classical examples of geometries on surfaces. This book is about these classical geometries and their close relatives which live on the same surfaces, have the same kinds of lines, and satisfy the same axioms as their classical counterparts.

The history of our geometries on surfaces starts with Hilbert constructing a first example of a nonclassical $\mathbf{R}^2$-plane, that is, a close relative of the Euclidean plane. Today, one century of research later, our book tries to summarize all major results about geometries on surfaces. This includes the following topics.

- Detailed descriptions of the classical geometries.
- The main methods of constructing nonclassical geometries.
- Classifications and characterizations of the geometries that are 'most homogeneous'. In particular, classifications of the various kinds of geometries with respect to the dimensions of their groups of auto-

morphisms which, in most cases, are known to be Lie groups of finite dimensions.

- Descriptions of the various geometries associated with a given geometry such as subgeometries, fix-geometries of automorphisms, and Lie geometries.
- Connections with other fields such as the theories of interpolation and approximation, convexity, differential geometry, topology, Lie groups, differential equations, oriented matroids, finite geometries, and higher-dimensional topological geometries.
- Open problems.

Putting Things into Perspective

Real Geometries: Most of the 'classical' types of geometries investigated in incidence geometry, such as affine and projective planes and circle planes, can be defined over all fields. The classical examples of geometries on surfaces are geometries associated with the real numbers. Just as the real numbers occupy a central position in the theory of fields, the classical geometries on surfaces occupy a central position in incidence geometry.

Topological Geometries: Our geometries on surfaces are topological geometries in that both the point and line/circle sets of our geometries carry natural topologies such that the geometric operations in the axioms that these geometries satisfy are continuous. As a result, the connecting line of two points in the Euclidean plane and the connecting circle of three points on a sphere depend continuously on the positions of the points on the surface. Among the topological incidence geometries, the geometries on surfaces are the ones that are most easily described and the ones that are best understood. A thorough understanding of these geometries is necessary for everybody interested in working in topological geometry.

Networks of Geometries: One of the most appealing features of geometries on surfaces is the tightly knit network of relationships that has been established between them. This network of relationships plus the wealth of our knowledge about the individual geometries identifies this cluster of geometries as one of the most well-charted sectors in incidence geometry. In other sectors of incidence geometry, in particular finite geometry, the most popular branch of incidence geometry today, the counterparts of our geometries are tied together into highly complex networks, the overall structure of which is very hard to discern because of the presence of a lot of 'noise', that is, sporadic geometries and links.

By stepping back and looking at the network of relationships between finite geometries of odd order through our 'topological filter', it is possible also to discern many of the fundamental links and results in our model cluster in the finite setting that would otherwise be very hard to see. We only remark that finite geometries of even order behave and are linked in ways that are very different from their odd order counterparts. Therefore it is not surprising that the correspondence between geometries on surfaces and finite geometries of even order is not very strong.

Geometric Combinatorics: The similarity of results in the finite and topological settings establishes a first close link between the theory of geometries on surfaces and combinatorics. There are two further strong links between these two disciplines. First, once a basic toolbox of techniques has been developed, these techniques can be employed in a very combinatorial manner to construct and derive results about geometries on surfaces. Second, finite subsets of circles of rank n geometries correspond to rank n pseudocircle arrangements that are fundamental objects in geometric combinatorics. In particular, finite subsets of lines in flat projective planes are pseudoline arrangements that have interpretations as important combinatorial structures such as oriented matroids, switching sequences, and primitive sorting networks.

Interpolation: The classical examples of our geometries also correspond to the classical objects around which other seemingly unrelated theories are built. The so-called classical tubular circle plane of rank n, for example, corresponds to the set of polynomials of degree at most $n-1$ over the reals, and the axiom of joining that this circle plane satisfies translates into the fact that this set of polynomials solves the Lagrange interpolation problem of order n. Similarly, every tubular circle plane of rank n corresponds to a set of continuous functions that solves the Lagrange interpolation problem of order n. In developing a theory of geometries on surfaces, we and our colleagues have also tried to provide a unifying topological foundation for the different kinds of disciplines that deal, in whatever guise, with geometries on surfaces.

Lie Groups: For many of the geometries on surfaces it is known that all their automorphisms are homeomorphisms of the surfaces, and their groups of automorphisms are finite-dimensional Lie transformation groups. This establishes an important connection between the theory of geometries on surfaces and the theory of Lie groups. In fact, the classification of finite-dimensional Lie groups is one of the most important tools used in the classification of geometries on surfaces.

Intended Audience

First and foremost our book is a reference book and people who will be interested in every single aspect of the book will most likely be mathematicians specializing in incidence geometry. Nevertheless, given that most of the geometries we are dealing with can be described, understood, visualized, and appreciated with only an advanced undergraduate's knowledge of topology and analysis, we have made every effort to keep large parts of the book accessible and appealing to as broad an audience as possible. Therefore, if you are a student interested in incidence geometry, try to develop a feeling for the way incidence geometries behave and interact by first studying the classical geometries over fields and geometries on surfaces. If you are a lecturer looking for new, beautiful, and easily accessible topics to spice up your introductory course on topology, analysis, the theory Lie groups, geometric combinatorics, or even interpolation theory, you will find that our book of geometries on surfaces has a lot to offer to you.

Contents and How to Read and Use This Book

We have tried to make this book as self-contained as possible. However, to get the most out of it and to arrive at a full understanding of every aspect of the theory of geometries on surfaces and topological geometry in general, we recommend that you use it side by side with a number of excellent books and survey articles. In particular, *Compact Projective Planes*, Salzmann et al. [1995], is the most comprehensive collection of results on topological projective planes and is a must-read for everybody interested in topological geometry. We also recommend the *Handbook of Incidence Geometry*, Buekenhout ed. [1995]. This book is an excellent up-to-date survey of modern incidence geometry. In particular, the last two chapters of this book deal with topological geometries. For a pictorial guide to geometries on surfaces see *A Geometrical Picture Book*, Polster [1998g]. For information about finite geometries also check out *Finite Geometries*, Dembowski [1968].

You should start by reading Chapter 1. In it we define the most important terms that we will be using in the rest of the book, describe the different kinds of geometries we are dealing with, and list many of their most important features in simple language.

In Chapter 2 we concentrate on flat linear spaces. These include the close relatives of the Euclidean plane and the projective plane over the real numbers. The results in this chapter have counterparts in the following chapters and many of these counterparts are based on the results in this chapter. In particular, we show that the line set of a flat linear space carries a natural topology, that automorphisms of such a geometry are continuous, and that the automorphism group of such a geometry is a finite-dimensional Lie transformation group. The logical continuation of this is to give an account of the classification of the flat linear spaces with respect to the dimensions of their automorphism groups. An excellent exposition of the classification of the flat projective planes whose automorphism groups have dimension at least 3 can be found in the book Salzmann et al. [1995]. Rather than copying all the relevant material in that book, we only summarize the main results derived in that book and extend them by many results, constructions, and classifications outside its scope. Highlights include the description of all flat projective planes whose automorphism groups are at least 2-dimensional, and all Möbius strip planes and $\mathbf{R}^2$-planes whose automorphism groups are at least 3-dimensional.

In Chapter 3 we focus on the spherical circle planes, that is, the relatives of the geometry of circles on a sphere. In it we demonstrate how to use the results about $\mathbf{R}^2$-planes developed in Chapter 2 to prove similar results for higher-rank flat circle planes. We also give a more detailed account of the group-dimension classification of spherical circle planes than we did for flat linear spaces. So, if you are interested in familiarizing yourself with typical arguments that are used in group-dimension classifications of topological geometries, here is a good spot to start. Some of the highlights in this chapter are the classification of the spherical circle planes whose automorphism groups are at least 4-dimensional, the Hering classification of these planes, and many new constructions of flat linear spaces from spherical circle planes.

In Chapters 4 and 5 we concentrate on the toroidal circle planes and the cylindrical circle planes. These include the flat Minkowski and Laguerre planes, respectively. Just like the spherical circle planes, the two types of circle planes we focus on in these chapters are of rank 3 and the results in these chapters are closely related to the results in Chapter 3. On the other hand, some new considerations enter the scene as both toroidal and cylindrical circle planes have nontrivial parallelisms. Included in these chapters are the classification of the flat Minkowski planes of group dimension at least 4 and the classification of the flat Laguerre planes of group dimension at least 5.

Chapter 6 is devoted to the theory of antiregular 3-dimensional generalized quadrangles as developed in *Topological Circle Planes and Topo-*

logical Quadrangles, Schroth [1995a]. This theory ties the flat Möbius, Laguerre, and Minkowski planes into a tight knot full of beautiful connections and unexpected relationships. In this chapter we give a summary of Schroth's theory and state the most important results about flat Laguerre, Möbius, and Minkowski planes that are best expressed in the language of generalized quadrangles. Also included in this chapter are the recently discovered connections between antiregular generalized quadrangles, circle planes, and semibiplanes.

In Chapter 7 we give an account of the connection between classical Lagrange and Hermite interpolation and higher-rank circle planes on surfaces.

In two appendices we summarize some basic results from analysis, topology, and the theory of Lie transformation groups that are frequently used in this book.

Acknowledgements

This book is based on the work of, and support by, many mathematicians: in particular, Helmut Salzmann under whose guidance topological geometry was created over the past 50 years, and Karl Strambach who introduced us to this beautiful branch of mathematics and encouraged us to pursue its study.

Particular thanks are due to Rainer Löwen and Andreas Schroth who read parts of the book and offered lots of helpful advice. Thanks are also due to Markus Stroppel who provided us with some urgently needed manuscripts.

The first author wishes to thank the Australian Research Council for its support through an Australian Postdoctoral Research Fellowship and Monash University for its support through a Logan Research Fellowship.

Finally, special thanks are due to Anu, Marina, and the kids for their support and patience throughout the writing of this book.

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