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Geometries for Pedestrians

1.1 Geometries of Points and Lines

In this book a *geometry* will usually consists of a nonempty *point set* and a nonempty *line set*, where a line is a subset of the point set containing at least three points and every point is contained in at least three lines. For example, the *Euclidean plane* is a geometry whose points are the points of the coordinate plane $\mathbf{R}^2$ and whose lines are the Euclidean lines. For historical reasons, lines are sometimes called *blocks* or *circles*. The lines of the geometry of circles on a sphere, for example, are really called circles. All geometries that we are interested in also satisfy a number of structuring *axioms*. In particular, both the Euclidean plane and the geometry of circles satisfy an 'axiom of joining'—in the Euclidean plane any two points are contained in exactly one line, and in the geometry of circles any three points are contained in exactly one circle. In fact, virtually all the geometries considered in this book satisfy an axiom of joining.

In most textbooks in our subject area lines and the sets of lines through points are required to contain only at least two elements. By restricting ourselves to *thick* geometries, it is possible to omit one axiom from each of the systems of axioms that we will encounter in this book.

> **Geometries Are Thick**
>
> All geometries in this book are *thick*, that is, every line contains at least three points and every point is contained in at least three lines.

The main disadvantage with our approach is that certain important graphs that are usually also regarded as geometries get excluded. In particular, the complete graph on four vertices, which is the smallest finite counterpart of the Euclidean plane, is not a geometry according to our definition. However, since we are concentrating on geometries on surfaces, we feel that the advantages of our approach outweigh its disadvantages.

By dealing with such basic geometries as the Euclidean plane, you will already have acquired a certain familiarity with many terms used in this book that should enable you to understand this chapter. For precise definitions of the most important terms used in this chapter and the rest of the book see Section 1.3 at the end of this chapter.

In the following we define some of the most fundamental types of geometries incidence geometers are dealing with: linear spaces, in particular projective planes and affine planes; the three types of Benz planes, that is, Möbius planes, Laguerre planes, and Minkowski planes; and orthogonal arrays. The classical examples of these geometries can be defined over many fields. The different geometries that correspond to the real numbers are the most important classical geometries on surfaces. We will also refer to projective planes, Benz planes, and certain maximal orthogonal arrays as *circle planes* since the lines/circles in the classical real examples of these planes are really topological circles.

A *linear space* is a geometry in which every two distinct points are contained in exactly one line. We start by defining projective and affine planes, two special types of linear spaces.

1.1.1 Projective Planes

A *projective plane* is a geometry that satisfies the following two axioms.

Axioms for Projective Planes

(P1) Two distinct points are contained in a unique line.

(P2) Two distinct lines intersect in a unique point.

The *classical examples* of projective planes are the projective planes over fields. Given a field F, this projective plane can be constructed as follows. The point set is the set of all 1-dimensional subspaces and the line set is the set of all 2-dimensional subspaces of the 3-dimensional vector space over F. In this model the two axioms are easily verified. Axiom P1 translates into the fact that two distinct 1-dimensional subspaces of a 3-dimensional vector space are contained in a uniquely determined 2-dimensional subspace. Similarly, Axiom P2 translates into the fact that two distinct 2-dimensional subspaces of a 3-dimensional vector space intersect in a 1-dimensional subspace. The classical projective plane over F is denoted by $\mathrm{PG}(2, F)$.

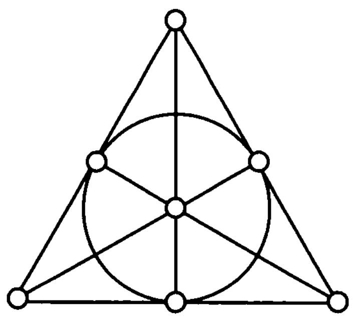

Fig. 1.1. The Fano plane

Figure 1.1 is a picture of the projective plane over the field with two elements. It has seven points and seven lines, every line contains three points and every point is contained in three lines. This projective plane is called the *Fano plane*. It is the unique smallest projective plane. A finite projective plane is of *order* n if and only if every one of its lines contains $n+1$ points and every one of its points is contained in $n+1$ lines. For example, the Fano plane is of order 2 and, in general, the classical projective plane over the field with q elements is of order $q+1$.

There are many ways in which the classical projective planes are characterized among the projective planes. One of the most famous such characterizations is via *Pappus' configuration*. Pappus' configuration is a geometry with nine points and nine lines, every line of which contains three points and every point of which is contained in three lines. Figure 1.2 is a picture of this configuration. Note that this diagram does not capture all the symmetries of this configuration. In fact, contrary to what the diagram suggests, none of the points and lines of the configuration is distinguished in any way.

It turns out that a projective plane is classical if and only if Pappus'

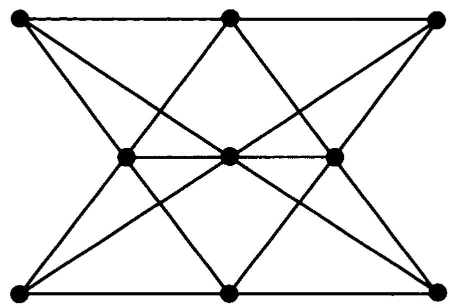

Fig. 1.2. Pappus' configuration

configuration *closes* in the projective plane; see p. 22 for a precise definition of the term 'closes'. In particular, for Pappus' configuration to close means that no matter how one draws the solid black parts of the configuration in Figure 1.2 using points and lines of the geometry, the three points that are contained in only two black lines each are always contained in a line of the geometry (the grey line). We only remark that the term 'Pappus' configuration closes' also encompasses some degenerate cases that arise when certain points of the configuration get identified.

Note that projective planes can also be defined over skewfields and that the projective planes constructed like this are precisely the ones in which *Desargues' configuration* closes; see Figure 1.3.

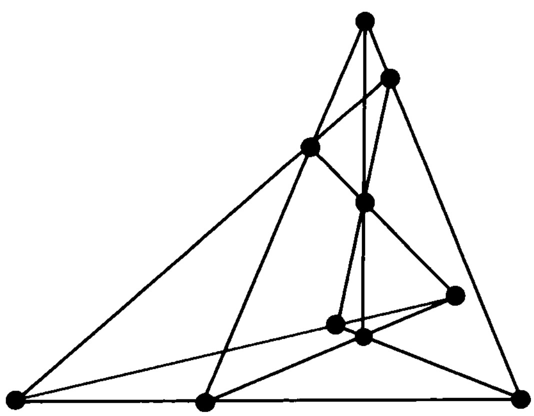

Fig. 1.3. Desargues' configuration

A projective plane is called a *Pappian* or *Desarguesian* projective plane if and only if Pappus' configuration or Desargues' configuration closes in it, respectively. Since every field is also a skewfield this implies that every Pappian projective plane is also Desarguesian. The converse is not true. However, for the projective planes that we are concentrating on in this book, namely the close relatives of the classical projective plane over the real numbers, the two notions coincide.

1.1.2 Affine Planes

By removing a line and all the points contained in it from a projective plane with at least four points on a line, we arrive at a geometry that satisfies the following axioms.

Axioms for Affine Planes

(A1) Two distinct points are contained in a unique line.

(A2) Given a line and a point not on this line, there is a unique line through the point that does not intersect the given line.

Every geometry that satisfies these axioms is called an *affine plane*. Two lines in an affine plane are called *parallel* if they coincide or do not intersect in a point. Axiom A2 implies that being parallel defines an equivalence relation on the line set. The equivalence classes of this equivalence relation are called *parallel classes*.

The affine plane we arrive at by removing a line from the classical projective plane associated with the real numbers is the Euclidean plane. All affine planes obtained from a classical projective plane by removing a line are isomorphic. Every affine plane has a unique *projective extension* to a projective plane. Starting with an affine plane, this projective plane can be constructed as follows. The points of the projective plane are the points of the affine plane plus its parallel classes of lines. The set of parallel classes is one of the lines of the projective plane. We construct the remaining lines by extending every line in the affine plane by the parallel class it is contained in.

A finite affine plane is of order n if its projective extension is of order n. Note that by removing a line and all its points from the projective plane of order 2, we arrive at the complete graph on four vertices. For completeness' sake, we call this graph the affine plane of order 2.

The affine plane associated with the classical projective plane over the field F can also be constructed as follows. The point set is $F \times F$, the lines are the verticals in the point set plus the graphs of all linear functions $F \to F$. In this plane the lines of a given slope form a parallel class. The verticals also form a parallel class.

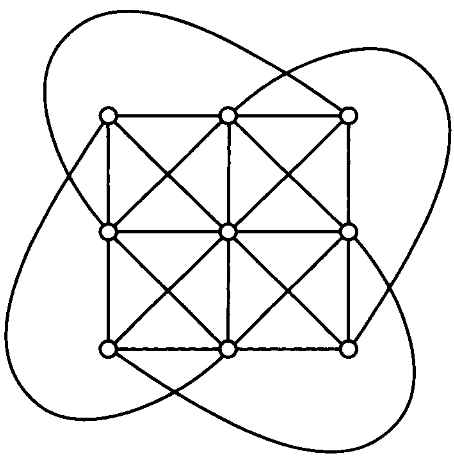

Fig. 1.4. The affine plane of order 3

Figure 1.4 shows a picture of the affine plane of order 3. In this plane every line contains three points and every point is contained in four lines. There are four parallel classes consisting of three lines each.

For comprehensive introductions to projective planes and their associated affine planes see Hughes–Piper [1973], Dembowski [1968], and Salzmann et al. [1995].

1.1.3 Benz Planes—the Original Circle Planes

When geometers talk about circle planes, they usually have the three different types of Benz planes in mind. In this book the term circle plane comprises infinitely many different types of geometries, not only the Benz planes. The following definition of Benz planes can be found in Steinke [1995].

A *Benz plane* is a geometry whose point set is equipped with one or two equivalence relations called *parallelisms*. Two points of a Benz plane are *parallel* if they are in relation with respect to one of the parallelisms. A parallelism is called *trivial* if two points are parallel if and only if they coincide. Furthermore, Benz planes satisfy the following axioms.

Axioms for Benz Planes

(B1) Three pairwise nonparallel points are contained in a uniquely determined circle.

(B2) Given a point p on a circle C and a point q not parallel to p, there is a uniquely determined circle that contains both points and *touches C geometrically*, that is, intersects C only in p or coincides with C.

(B3) Parallel classes with respect to a nontrivial parallelism and circles intersect in a unique point.

(B4) Parallel classes with respect to different nontrivial parallelisms intersect in a unique point.

If a Benz plane has two different nontrivial parallelisms, it is a *Minkowski plane*. If it has only one nontrivial parallelism, it is a *Laguerre plane*. In this case Axiom B4 does not apply. If it has only a trivial parallelism, both Axioms B3 and B4 do not apply, 'nonparallel' translates into 'distinct', and the Benz plane is a *Möbius plane*.

The classical examples of Möbius, Laguerre, and Minkowski planes arise as the geometries of nontrivial plane sections of elliptic quadrics, elliptic cones, and hyperbolic quadrics, respectively, in the 3-dimensional projective spaces over fields.

In the real case that we are most interested in two of the classical geometries live in a real Euclidean 3-space. The classical real Möbius plane is just the geometry of circles on a sphere, and the classical real Laguerre plane is the geometry of nontrivial plane sections of a cylinder. The set of generator lines on this cylinder are the parallel classes of points. The classical real Minkowski plane minus one of its circles and all points on this circle is the geometry of nontrivial plane sections of a (one-sheeted) hyperboloid. The two sets of parallel classes correspond to the disjoint partitions of the hyperboloid into Euclidean lines embedded in the hyperboloid.

The *derived plane* at a point p of a geometry $\mathcal{G}$ whose point set carries one or more parallelisms has all the points of $\mathcal{G}$ not parallel to p as points. Its lines are all lines of $\mathcal{G}$ through p that have been punctured at p plus all nontrivial parallel classes not containing p that have been punctured at all points parallel to p. It turns out that a geometry having one or two parallelisms and satisfying Axioms B3 and B4 is a Benz plane if and only if the derived planes at all its points are affine planes.

A finite Benz plane is of order n if all its circles contain $n+1$ points. The derived planes of a Benz plane of order n are affine planes of order n. A derived plane of a classical Benz plane over the field F is isomorphic to the classical affine plane over F. The derived planes of the geometry of circles on a sphere are isomorphic to the Euclidean plane.

The picture of the affine plane of order 3 in Figure 1.4 is also a picture of the unique Minkowski plane of order 2. Here the parallel classes are the verticals and horizontals in the grid and the circles are the remaining six lines in the affine plane.

1.1.4 Orthogonal Arrays

An *orthogonal array of rank n* is a geometry whose point set carries one nontrivial parallelism. Its point set is of the form $C \times E$, where C and E are disjoint sets, with $|C| \geq n$ and $|E| \geq 2$. Furthermore, two of its points $(a, b), (c, d) \in C \times E$ are parallel if and only if $a = c$, and it satisfies the following axioms.

Axioms for Orthogonal Arrays

(O1) Any n pairwise nonparallel points are contained in a uniquely determined circle.

(O2) A parallel class and a circle intersect in a unique point.

Given an orthogonal array O of rank n, we can identify C and E with a circle and a parallel class, respectively. Because of Axiom O2, circles are graphs of functions $C \to E$.

Starting with the polynomials of degree at most $n-1$ over a field F with at least $n \geq 3$ elements, we define a geometry $\mathrm{Poly}(n, F)$ whose point set is $F \times F$ and whose circles are the graphs of the polynomials. Keeping in mind the interpolating property of the set of polynomials under consideration, it is clear that this geometry is an orthogonal array of rank n. The parallel classes are the verticals in the point set. For example, the geometry $\mathrm{Poly}(2, \mathbf{R})$ is the Euclidean plane in which the verticals lines have been turned into parallel classes.

By removing one parallel class together with all the points contained in it from an orthogonal array of rank n, we are left with an orthogonal array of the same rank as long as $|C| \geq \max\{4, n+1\}$. An orthogonal array of rank n is called *maximal* if it does not arise from an orthogonal array of the same rank in this manner. The polynomial geometry $\mathrm{Poly}(n, F)$ is not maximal since it can be extended by a *parallel class at infinity* $\{\infty\} \times F$ to a larger orthogonal array $\overline{\mathrm{Poly}}(n, F)$. Here the point (∞, a) extends the circles that correspond to polynomials in which a is the coefficient of x^{n-1}.

The geometries $\mathrm{Poly}(n, F)$ and $\overline{\mathrm{Poly}}(n, F)$ are the *classical orthogonal arrays*. For example, the geometry $\overline{\mathrm{Poly}}(3, \mathbf{R})$ is isomorphic to the Laguerre plane over the real numbers. Clearly, every Laguerre plane is an orthogonal array of rank 3.

We remark that some of the terminology in the literature on finite orthogonal arrays differs from ours. For example, the word 'strength' is used instead of 'rank' and the orthogonal arrays we defined here are further said to be of index 1; see Rao [1947], Bush [1952], and Colbourn–Dinitz [1996] Part 2.

1.2 Geometries on Surfaces

In this section we first have a quick look at flat linear spaces, that is, geometries that are closely related to the Euclidean plane. Following this, we give an overview of the most important geometries on surfaces. These are geometries whose lines or circles are topological circles. We call these geometries *flat circle planes*. They form the backbone of the theory of geometries on surfaces, and most other types of geometries that we will come across in this book have representatives that can be easily derived from flat circle planes.

We also try to give you a feel for what geometries on surfaces are all about by describing some of their basic features, connections with the theories of interpolation and convexity, and parts of the network of relationships that turns the different types of geometries on surfaces into a larger whole.

1.2.1 (Ideal) Flat Linear Spaces

Ideally, we would like to develop a general theory of the *linear spaces on surfaces*, that is, linear spaces whose point sets are surfaces and all of whose lines are **R**-lines and/or **S**-lines, that is, closed subsets of the point sets homeomorphic to **R** and/or the circle $\mathbf{S}^1$, respectively. All known linear spaces on surfaces are *flat linear spaces*, that is, linear spaces on surfaces whose line sets can be equipped with topologies such that the operations of joining two points by a line and intersecting two lines in a point are continuous on their domains of definition. Furthermore, in a flat linear space the set of pairs of distinct intersecting lines is required to be open; see Section 2.5. Flat linear spaces live on surfaces homeomorphic to $\mathbf{R}^2$, the real projective plane, and the Möbius strip and are referred to as $\mathbf{R}^2$-planes, flat projective planes, and Möbius strip planes, respectively. We suspect that the flat linear spaces may well encompass all linear spaces on surfaces. However, it has not even been proved that the three surfaces mentioned above are the only point sets that a linear space on a surface can live on; see Problem 2.11.2.

We do know that the $\mathbf{R}^2$-planes and flat projective planes are precisely the linear spaces on surfaces that are homeomorphic to $\mathbf{R}^2$ and the real projective plane, respectively. In $\mathbf{R}^2$-planes all lines are **R**-lines and in flat projective planes all lines are **S**-lines. Möbius strip planes are of mixed type, that is, have both **R**- and **S**-lines. We do not know whether the Möbius strip planes are the only linear spaces on surfaces homeomorphic to the Möbius strip; see Problem 2.11.1.

Given a flat projective plane with point set P, examples of $\mathbf{R}^2$-planes and Möbius strip planes arise as the restrictions of this flat projective plane to certain open subsets of P. However, not all $\mathbf{R}^2$-planes and Möbius strip planes arise in this manner. The restriction to the complement of a line is called a flat affine plane and the restriction to the complement of a point is called a point Möbius strip plane. Both flat affine planes and point Möbius strip planes can be considered as special representations of the flat projective plane they are associated with since this flat projective plane can be reconstructed from both geometries in a unique manner.

Apart from being models of flat projective planes, point Möbius strip planes also play an important role in the general theory of tubular circle planes as they are in one-to-one correspondence with the tubular circle planes of rank 2; see Subsection 1.2.2 and Chapter 7.

We do not want to start our investigations with the above abstract