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Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces

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Preface

These notes are based on lectures given in 1994 by the first author at a Summer School in Tuczno (Poland) and at the University of Metz.

The purpose is to give a quick introduction to ergodic theory, to study interesting examples as illustration and to present some recent and spectacular developments in topological dynamics of group actions. More precisely, the focus will be on the following two types of systems of a geometrical–algebraic nature:

The geodesic flow on the unit tangent bundle of a locally symmetric space and unipotent actions on homogeneous spaces. Classical examples are the geodesic flow and the horocyclic flow on the unit tangent bundle of a compact Riemann surface of constant negative curvature. These flows are among the most studied dynamical systems. Of particular interest are their ergodic (or mixing) properties and the asymptotic behaviour of their orbits.

Unipotent actions on homogeneous spaces enjoy remarkable regularity properties. A striking illustration of this regularity is Hedlund's minimality theorem: for any lattice Γ in $G = \mathrm{SL}(2,\mathbb{R})$, any orbit in the homogeneous space $\Gamma \backslash G$ under a unipotent subgroup of $\mathrm{SL}(2,\mathbb{R})$ is either periodic or dense. Such actions have close connections to problems in Number Theory. For instance, one of the most spectacular application of unipotent actions is the solution in 1987 by Margulis of Oppenheim's conjecture which was open for more than 40 years (see [Ma3]). Margulis' proof is based on the study of the dynamical properties of the action of unipotent one-parameter subgroups of $\mathrm{SO}(2,1)$ on $\mathrm{SL}(3,\mathbb{R}) / \mathrm{SL}(3,\mathbb{Z})$.

In this context, Raghunathan as well as Dani and Margulis formulated in the eighties general conjectures about closure and distribution of orbits of unipotent flows. These so-called Raghunathan conjectures were settled at the beginning of the nineties by Ratner ([Ra2]–[Ra6]).

In theses notes, the emphasis will be put on the group theoretical point of view, and, whenever possible, on the rôle of unitary group representations. So, ergodicity and mixing will be seen to be special instances of the powerful Howe–Moore's theorem about the asymptotic behaviour of matrix coefficients of unitary representations of semisimple groups.

Group representations enter the picture in the following way. A compact Riemann surface Σ of genus $g \geq 2$ has the real hyperbolic plane $\mathbf{H}^2$ as the universal covering space and may therefore be identified with the orbit space $\Gamma \backslash \mathbf{H}^2$ for a discrete cocompact subgroup of $\mathrm{PSL}(2,\mathbb{R})$. In this way, the geodesic flow and the horocycle flow on the unit tangent bundle $T^1(\Sigma)$ of Σ may be identified with flows on the homogeneous space $\Gamma \backslash \mathrm{PSL}(2,\mathbb{R})$ given by right translations by elements from the subgroup A and N of the diagonal and of the unipotent upper triangular matrices in $\mathrm{PSL}(2,\mathbb{R})$, respectively. Now, the constant functions on $T^1(\Sigma)$ are certainly invariant under A (or N). Ergodicity means that there are no other

A-invariant (or N-invariant) functions in $L^2(T^1(\Sigma))$, a space carrying a natural unitary representation of $\mathrm{PSL}(2,\mathbb{R})$.

This point of view has first been adopted by Gelfand and Fomin [GF] in the fifties. Their methods were based on the explicit description of the unitary dual of $\mathrm{SL}(2,\mathbb{R})$. In fact, such a description is not necessary for that purpose (and not always available for other groups). Mautner [Mau] treated the geodesic flow for all locally symmetric spaces in this spirit. Subsequently, Moore [Mo1] proved his celebrated ergodicity theorem, a very general result valid for all semisimple Lie groups. This, in turn, is a special case of the Howe–Moore theorem about the vanishing at infinity of matrix coefficients of unitary representations.

Here are the essential features of these notes:

(i) We give a complete proof of Howe–Moore's vanishing theorem alluded to above. Using root theory, we first reduce to the case of $\mathrm{SL}(2,\mathbb{R})$. For this group, we follow the elementary and elegant proof from [HT]. Moore's ergodicity theorems are deduced from the vanishing theorem.

(ii) We give Mautner's characterization of ergodicity of the geodesic flow for locally symmetric spaces.

(iii) Following Ratner [Ra7], a proof of Raghanuthan's conjectures for the case $\mathrm{SL}(2,\mathbb{R})$ is given. More precisely, the invariant measures under the horocyclic flow are classified. This classification was first obtained by Furstenberg [Fu1] in the case of cocompact lattices and by Dani [Da1] for general lattices. Moreover, equidistribution of the orbits of the horocyclic flow is established. This is Dani–Smillie's equidistribution theorem [DS] which may be viewed as a quantitative refinement of the minimality theorem of Hedlund mentioned above. Ratner's results which are valid in much greater generality are beyond the scope of these notes. However, the case of $\mathrm{SL}(2,\mathbb{R})$ presented here already contains many crucial ideas involved in the general situation.

(iv) We give an elementary, complete proof of Oppenheim's conjecture. This conjecture says that, if Q is a nondegenerate indefinite form on $\mathbb{R}^n$, $n \geq 3$, and if Q is not a multiple of a rational form, then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$. Our treatment is based on the article [DM3] by Dani and Margulis. We tried to emphasize the common features of this proof with Hedlund's minimality theorem

Some other features are worth mentioning. In view of the proof of Oppenheim's conjecture, we discuss the construction of Siegel sets in $SL(n,\mathbb{R})$, proving Mahler's compactness criterion, and give the proof of Margulis' lemma about recurrence of orbits of unipotent one-parameter groups in $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$. As an application of mixing, we establish – following [EM] – an asymptotic formula for the number of lattice points in a ball in the hyperbolic plane. We also discuss Mozes' result about mixing of all orders of actions of semisimple Lie groups, as well as Ledrappier's counterexample of a mixing action of $\mathbb{Z}^2$ which is not mixing of all orders.

Description of the contents

Here is an overview about the contents of these notes:

Chapter I is devoted to some general facts about ergodic systems: definitions, first examples, classical theorems such as Poincaré's recurrence theorem, the ergodic theorems of von Neumann and of Birkhoff. We borrowed from M.S. Keane ([BKS]) a particularly elegant proof for Birkhoff's ergodic theorem. Special attention is paid to stating the different notions of mixing in representation theoretic terms. Finally, we present some facts about the existence of invariant measures for group actions and study the question of when ergodic measures are supported on orbits.

Chapter II deals with our first main example, the geodesic flow on a locally symmetric space. We treat in great detail the standard facts about Fuchsian groups and their fundamental domains (Dirichlet regions). We describe the geodesic flow on a Riemann surface of negative constant curvature and, more generally, on a locally symmetric Riemannian space in group theoretic terms. Mautner's rank one criterion for ergodicity is discussed.

In Chapter III, we give the complete proof of Howe–Moore's vanishing theorem. We discuss some examples in connection with Moore's duality theorem. As an application, we give the asymptotic formula for the lattice point problem in the hyperbolic plane. We discuss Ledrappier's counterexample as well as Mozes' result on mixing of all orders of Lie group actions.

Chapter IV deals with the horocycle flow. We first discuss Hedlund's minimality theorem in the case of a compact surface. Following [Ra7], we then prove the classification result of invariant measures as well as Dani–Smillie's equidistribution theorem.

In Chapter V, we discuss in length the construction of Siegel sets in $SL(n,\mathbb{R})$, showing that $SL(n,\mathbb{Z})$ is a lattice and proving Mahler's compactness criterion. This chapter contains also the complete proof of Margulis' lemma.

In Chapter VI, a strengthening – due to Dani and Margulis – of Oppenheim's conjecture is proved according to which the set of values taken by an indefinite irrational form in $n \geq 3$ variables on the primitive integer vectors in $\mathbb{Z}^n$ is dense.

Prerequisites

Concerning the prerequisites, we only assume the reader to be familiar with the elementary facts from functional analysis, measure theory and Lie theory. For convenience, more elaborate notions and results – such as the Cartan decomposition of semisimple Lie groups – are shortly recalled when they are needed. In the concrete examples we have in mind, such results correspond usually to elementary facts which are easy to prove. These notes are intended to be accessible for mathematicians and graduate students with various backgrounds.

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