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Chapter I

Ergodic Systems

Ergodic theory may be viewed as the study of measure (or, more generally, measure class) preserving actions of groups (or semigroups) on measure spaces.

The main examples to be treated throughout these notes arise as follows. Let G be a locally compact group, and let H, L be closed subgroups of G. The homogeneous space G/H carries a unique G-invariant measure class. Now, L acts on G/H by left translations. An interesting and important problem is to study, for specific G, H, L this action of L on G/H from a measure theoretic point of view. Usually, H is a lattice in G (see Chap. II, §2) so that G/H carries a G-invariant probability measure. So, we shall almost always deal with measure preserving actions on a probability space.

This chapter is a quick introduction to ergodic theory. We discuss mainly material which is relevant for later chapters.

Our exposition is incomplete as several important topics, such as entropy, have been omitted. Section 1 contains some standard examples of ergodic actions. In Section 2, ergodicity is formulated in terms of unitary group representations (the so-called Koopmanism). The classical ergodic theorem of von Neumann is proved and M. Keane's elegant proof of Birkhoff's ergodic theorem is reproduced. Moreover, strong mixing and weak mixing are introduced and discussed from the point of view of unitary representations. In Section 3, we state the theorem about the decomposition of general measure preserving group actions into ergodic pieces. We discuss also the existence of invariant measures for a continuous transformation of a compact space, as well as the problem of uniqueness of ergodic measures.

§1 Examples and Basic Results

Let G be a locally compact second countable group with identity e. Let (X, μ) be a measurable space with a σ-finite measure μ. Recall that a measure μ on a measure space X is *finite* if $\mu(X) < \infty$ and that μ is σ-*finite* if X is a countable union of measurable subsets of finite measure.

In what follows, all measures are assumed to be σ-finite.

1.1 Definition. An *action* of G on X is a measurable mapping

$$G \times X \to X\,,\ (g,x) \mapsto g\,x$$

with the following two properties:
(i) $g_1\,(g_2\,x) = (g_1\,g_2)\,x\,,\ e\,x = x$, for all $g_1\,,\ g_2 \in G, x \in X$, and
(ii) G preserves the measure class of μ, that is, for all measurable subsets A of X and all $g \in G$, one has $\mu(gA) = 0$ if and only if $\mu(A) = 0$. One says that μ is *quasi-invariant.*

The action of G is *ergodic* if there are no non-trivial invariant subsets of X, that is, if the following holds: if A is a measurable and G-invariant subset of X, then $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Usually, the measure μ on X will even be *invariant* under G, that is, $\mu(gA) = \mu(A)$ for all measurable subsets A of X and all $g \in G$. However, there are interesting examples where this is not the case such as the natural action of $G = \mathrm{SL}(2,\mathbb{Z})$, the group of integer $2{\times}2$ matrices with determinant 1, on the real projective line $\mathbb{RP}^1$ (see Chap. III, Examples 2.9).

Recall that every locally compact group G has a non-zero locally finite Borel measure which is invariant under left translations and that such a measure, called *Haar measure*, is unique, up to a constant factor (see, e.g., [Wei] or [HeR]). A Borel measure μ on a locally compact space X is *locally finite* or *regular* if $\mu(K) < \infty$ for every compact subset K of X.

Very often, it is more convenient to work with functions instead of subsets. Let G be a group acting on a measure space (X,μ). A measurable function $f : X \to \mathbb{R}$ is *essentially G-invariant* if, for any $g \in G$, one has $f(gx) = f(x)$ for μ-almost all $x \in X$. The function $f : X \to \mathbb{R}$ is *G-invariant* if $f(gx) = f(x)$ for all $g \in G$ and all $x \in X$.

The following useful lemma shows that an essentially G-invariant function on X agrees almost everywhere with a G-invariant function (see [Zi], 2.2.16 Lemma). (Observe that this is obvious when G is countable.)

1.2 Lemma. *Let G be a locally compact second countable group acting on a σ-finite measure space (X,μ). Let $f : X \to \mathbb{R}$ be a measurable essentially G-invariant function on X. Then there exists a measurable G-invariant function $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f} = f$ almost everywhere on X.*

Proof Replacing f by $\phi \circ f$ for a homeomorphism $\phi : \mathbb{R} \to (0,1)$, we may clearly assume that $f(X) \subseteq (0,1)$.

Fix a probability measure λ on G in the measure class of the Haar measure of G. The subset

$$Q = \{(g,x) \in G \times X \,|\, f(gx) \neq f(x)\}$$

of $G \times X$ is measurable and, for all $g \in G$,

$$\int_X \chi_Q(g,x)d\mu(x) = \mu(\{x \in X \,|\, f(gx) \neq f(x)\}) = 0,$$

where χ_Q denotes the characteristic function of Q. Let X_0 be the set of all $x \in X$ for which

$$\int_G \chi_Q(g,x)d\lambda(g) = \lambda(\{g \in G \,|\, f(gx) \neq f(x)\}) = 0.$$

As G and X are σ-finite measure spaces, Fubini's theorem applies and shows that X_0 is measurable and that $\mu(X \setminus X_0) = 0$.

Let X_1 be the subset of all $x \in X$ for which the mapping $g \mapsto f(gx)$ is constant almost everywhere on G (that is, constant outside a subset of G of measure zero). Clearly, X_1 is G-invariant. Define

$$F : X \to (0,1), \quad F(x) = \int_G f(gx)d\lambda(g)$$

Then, again by Fubini's theorem, F is measurable. Observe that F agrees with f on the set X_0. The set

$$\{(g,x) \in G \times X \,|\, f(gx) \neq F(x)\}$$

is measurable. As above, Fubini's theorem shows that the set X_2 of all $x \in X$ for which

$$\lambda(\{g \in G \,|\, f(gx) \neq F(x)\}) = 0,$$

is measurable and that $\mu(X \setminus X_2) = 0$. Now, it is clear that $X_2 = X_1$. Fix any $a \in (0,1)$, and define $\tilde{f} : X \to (0,1)$ by $\tilde{f}(x) = F(x)$ if $x \in X_1$ and $\tilde{f}(x) = a$ if $x \in X \setminus X_1$. Then $\tilde{f} : X \to (0,1)$ is a measurable G-invariant function and $\tilde{f} = f$ on X_0. □

The following rephrasis of ergodicity will allow us in §2 to give a representation theoretic formulation of ergodicity when μ is invariant and finite.

1.3 Theorem. *Let G be a locally compact second countable group acting on a σ-finite measure space (X,μ). Then the following are equivalent:*

(i) *The action of G is ergodic;*

(ii) *If $f : X \to \mathbb{R}$ is measurable and essentially G-invariant, then f is constant almost everywhere.*

Proof Let A be a G-invariant measurable subset of X. Then χ_A, the characteristic function of A, is G-invariant. Hence, (ii) implies (i).

To show that (i) implies (ii), assume that G acts ergodically on X. Let f be a measurable and essentially G-invariant function on G. By the previous lemma, we may assume that f is G-invariant. For $k \in \mathbb{Z}$, and $n \in \mathbb{N}$, define

$$X(k,n) = \left\{ x \in X | \ \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\} .$$

Clearly, $X(k,n)$ is measurable and G-invariant. Hence, by ergodicity,

$$\mu(X(k,n)) = 0 \quad \text{or} \quad \mu(X \setminus X(k,n)) = 0.$$

Fix $n \in \mathbb{N}$. Then

$$X = \biguplus_{k \in \mathbb{Z}} X(k,n) \quad \text{(disjoint union)}.$$

Hence, there exists $k_n \in \mathbb{Z}$ so that

$$\mu(X \setminus X(k_n, n)) = 0.$$

Define

$$Y := \bigcap_{n=1}^{\infty} X(k_n, n) = X \setminus \left(\bigcup_{n=1}^{\infty} X \setminus X(k_n, n) \right).$$

Then $\mu(X \setminus Y) = 0$, and f is constant on Y. □

1.4 Examples.

(i) **Classical mechanics.**

Examples of measure preserving actions arise in classical mechanics (see [AA]). The motion of k particles in the state space Ω, governed by Hamilton's equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (1 \leq i \leq k)$$

defines a measure preserving action

$$\mathbb{R} \times \Omega \to \Omega, \ (t, \omega) \mapsto \varphi_t(\omega)$$

of $\mathbb{R}$ on Ω, where Ω, a subset of $\mathbb{R}^{6k}$, with coordinates $(p,q) \in \mathbb{R}^{6k} = \mathbb{R}^{3k} \times \mathbb{R}^{3k}$, is equipped with Liouville measure $dp\,dq$ and $\varphi_t(\omega)$ denotes the state of the system at time t starting from $\omega \in \Omega$ at time 0. (The invariance of $dp\,dq$ follows from Liouville's theorem; see Example 1.9 (i) below.)

An interesting question is the so-called *ergodic hypothesis of Bolzmann*: let Ω_E be a level surface for the energy and let $f : \Omega_E \to \mathbb{R}$ be a measurable function. Is the time average of f equal to its space average, that is, does

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi_t(\omega))\, dt = \int_{\Omega_E} f \; dp\, dq$$

hold for $\omega \in \Omega_E$?
In particular, does, for $A \subseteq \Omega_E$, the limit $\lim_{T\to\infty} \frac{r(T)}{T}$ agree with the volume of A, where $r(T)$ is the measure of the set

$$\{s \mid 0 \le s \le T \text{ such that } \varphi_s(\omega) \in A\}$$

(ii) Let $X = \mathbb{T} = \{z \in \mathbb{C} |\ |z| = 1\}$ be the one-dimensional torus with normalized Lebesgue measure μ. Take $\alpha \in \mathbb{R}$ and define

$$\varphi : \mathbb{T} \to \mathbb{T}\,, \ \varphi(z) = e^{2\pi i \alpha} z\ .$$

Clearly, φ preserves μ. Hence φ defines an action of the integers $\mathbb{Z}$ on $\mathbb{T}$ via

$$n \cdot z = \varphi^n(z)\ , \ \forall n \in \mathbb{Z}.$$

Two cases may occur:
- If $\alpha \in \mathbb{Q}$, then $\varphi^N = \mathrm{id}_{\mathbb{T}}$ for some $N \in \mathbb{N}$. It is clear that φ is not ergodic.
- If $\alpha \notin \mathbb{Q}$, then φ is ergodic.

Indeed, let A be a measurable and φ-invariant subset of $\mathbb{T}$, and let χ_A be its characteristic function. Let a_n be the n-th Fourier coefficient of χ_A. The n-th Fourier coefficient of the characteristic function of

$$e^{-2\pi i\alpha} A = \varphi^{-1}(A)$$

is $a_n e^{2\pi i n\alpha}$. Hence, by invariance of A,

$$a_n e^{2\pi i n\alpha} = a_n, \quad \forall n \in \mathbb{Z}.$$

Since α is irrational, this implies

$$a_n = 0, \quad \forall n \in \mathbb{Z}, n \neq 0.$$

As χ_A coincides, as L^2-function, with its Fourier series

$$z \mapsto \sum_{n=-\infty}^{\infty} a_n z^n,$$

this shows that χ_A is constant almost everywhere. Hence, $\mu(A) = 0$ or $\mu(A) = 1$.

(iii) The group $G = \mathrm{SL}(n,\mathbb{Z})$ of all $n \times n$ matrices with integer entries and with determinant 1 acts on $\mathbb{R}^n$ in a natural way, preserving the lattice $\mathbb{Z}^n$. Hence, G acts (as a group of automorphisms) on the n-dimensional torus

$$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n.$$

Moreover, G preserves the Lebesgue measure μ on $\mathbb{T}^n$. We claim that each matrix γ in $\mathrm{SL}(n,\mathbb{Z})$ which has no root of unity as eigenvalue acts ergodically on $\mathbb{T}^n$. As in the above example, the proof will use Fourier analysis. Indeed, let A be a measurable γ-invariant subset of $\mathbb{T}^n$, and, for $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, let

$$a_k = \int_{\mathbb{T}^n} \chi_A(z) e^{-2\pi i z.k} dz$$

denote the corresponding Fourier coefficient of χ_A, where

$$z.k := z_1 k_1 + \cdots + z_n k_n$$

for $z = (z_1, \ldots, z_n) \in \mathbb{T}^n$ and dz is the normalized Lebesgue measure on $\mathbb{T}^n$. The Fourier coefficient of $\chi_{\gamma A}$ corresponding to k is $a_{\gamma^t k}$. Hence, by invariance of A,

$$a_k = a_{\gamma^t k}, \quad \forall k \in \mathbb{Z}^n.$$

But, for any $k \in \mathbb{Z}^n$, $k \neq 0$, the set $\{\gamma^n k \mid n \in \mathbb{Z}\}$ is infinite, since γ has no root of unity as eigenvalue. As $\sum_{k \in \mathbb{Z}^n} |a_k|^2$ is finite, a_k has to be zero for $k \in \mathbb{Z}^n \setminus \{0\}$. As above, this shows that χ_A is constant.

Example (iii) generalizes as follows to an action of a group G on a compact abelian group X. We recall a few facts about the duality theory of a compact abelian group X. (For more details, see [HeR], Chap. VI.) The *dual group* $\widehat{X}$ of X is the group of all continuous unitary characters $\sigma : X \to \mathbb{T}$. It is discrete for the topology of uniform convergence.

Let X be equipped with its normalized Haar measure dx. The Fourier coefficients of a function f in $L^1(X)$ are the complex numbers

$$\hat{f}(\sigma) := \int_X f(x)\overline{\sigma(x)} dx, \quad \forall \sigma \in \widehat{X}.$$

The Fourier transform

$$L^2(X) \to \ell^2(\widehat{X}), \quad f \mapsto \hat{f} = (\hat{f}(\sigma))_{\sigma \in \widehat{X}}$$

is a Hilbert space isometry (this is the Plancherel Theorem).

If a group G acts by automorphisms on X, then the dual action of G

on $\widehat{X}$ is of course given by

$$g\sigma(x) = \sigma(g^{-1}x), \quad \forall g \in G, \sigma \in \widehat{X}, x \in X.$$

Observe that the (normalized) Haar measure μ on X is automatically preserved by G. Indeed, let φ be an automorphism of X, and let $\varphi_*(\mu)$ be the image of μ under φ. Then, by unicity of Haar measure, $\varphi_*(\mu)$ is a multiple of μ. On the other hand, $\varphi_*(\mu)(1) = \mu(1)$. This shows that $\varphi_*(\mu) = \mu$.

1.5 Proposition. *Let G be a group acting as a group of automorphisms of a compact abelian group X. Let X be equipped with its normalized Haar measure. Then the action of G on X is ergodic if and only if, except the trivial character $\{1_X\}$, all the G-orbits for the dual action on $\widehat{X}$ are infinite.*

Proof The proof follows *mutatis mutandis* the one given in example (iii) above.

Indeed, let A be a G-invariant measurable subset of X. For $f \in L^2(X)$ and $g \in G$, let $T_g f$ be the function on X defined by

$$T_g f(x) = f(gx), \qquad \forall x \in X.$$

The Fourier coefficient of $T_g f$ at $\sigma \in \widehat{X}$ is

$$\widehat{T_g f}(\sigma) = \hat{f}(g\sigma).$$

Take now $f = \chi_A$, the characteristic function of A. Since A is G-invariant, the above shows that the Fourier transform $\hat{f}$ of f is constant on the G-orbits in $\widehat{X}$. Observe that $\hat{f} \in \ell^2(\widehat{X})$ and that f is constant if and only if $\hat{f}(\sigma) = 0$ for all $\sigma \in \widehat{X}$, $\sigma \neq 1_X$. This proves the claim. □

An interesting feature of the above examples is the use of the powerful Fourier analysis methods in order to establish ergodicity. This is not an accidental fact. Indeed, we shall see in the next section how ergodicity fits in the more general framework of unitary group representations.

1.6 Exercise. Let $Y = \{0, 1, \ldots, k-1\}$, $k \in \mathbb{N}$, with the probability measure p,

$$p(i) := 1/k \qquad \forall 0 \leq i \leq k.$$

Let $X = Y^{\mathbb{Z}}$ be the product space $\prod_{n\in\mathbb{Z}} Y$, with the product measure μ. Let

$$\varphi : X \to X$$

be the *Bernoulli-shift* defined by

$$\varphi\big((x_n)_{n\in\mathbb{Z}}\big) = (x_{n+1})_{n\in\mathbb{Z}}.$$

Then φ preserves μ. Show that φ is ergodic .

Hint: Write $Y = \mathbb{Z}/k\mathbb{Z}$, and view X as a compact abelian group. Then μ is the normalized Haar-measure on Y and φ is an automorphism of X. More general shifts are obtained when p is an arbitrary probability measure on Y; concerning their egodicity, see [Wal], Theorem 1.12.

An interesting phenomenon in the case of a finite invariant measure is recurrence.

1.7 Theorem (Poincaré's Recurrence Theorem). *Let (X,μ) be a finite measure space, and let $\varphi : X \to X$ be a measure preserving mapping. Let A be a measurable subset of X. Then almost every point $x \in A$ is infinitely recurrent with respect to A, that is, the set*

$$\{n \in \mathbb{N} |\ \varphi^n x \in A\}$$

is infinite.

Proof Fix $m \in \mathbb{N}$, and set

$$B_m := \{x \in A |\ \varphi^n x \notin A, \forall n \geq m\}.$$

Since $B_m = A \setminus \bigcup_{j\geq m} \varphi^{-j}(A)$, the set B_m is measurable. It is clear that $B_m \cap \varphi^{-j}(B_m) = \emptyset$ for each $j \geq m$. Hence,

$$\varphi^{-j}(B_m) \cap \varphi^{-k}(B_m) = \emptyset$$

for $j-k \geq m$. Thus $\{\varphi^{-jm}(B_m)\}_{j\in\mathbb{N}_0}$ is a disjoint sequence of measurable sets. By invariance of μ, all these sets have the same measure. Since $\mu(X)$ is finite, this implies that $\mu(B_m) = 0$. □

Very often, the measure space X comes with a (natural) topology. In that case, we have the following topological version of Poincaré's Recurrence Theorem.

1.8 Corollary (Poincaré's Recurrence Theorem – Topological Version). *Let (X,d) be a separable metric space, μ a finite Borel measure on X and $\varphi : X \to X$ a measure preserving mapping. Then μ-almost every point $x \in X$ is recurrent, that is, there is a sequence of integers $n_1 < n_2 < \cdots$ such that*

$$\lim_{k\to\infty} \varphi^{n_k} x = x.$$

Proof Let $\{U_n\}_n$ be a countable basis for the topology of X, and let $\widetilde{X}$ be the collection of all recurrent points in X. Then

$$X \setminus \widetilde{X} = \bigcup_n Y_n,$$

where Y_n is the set of all points $x \in U_n$ which are not infinitely recurrent with respect to U_n. By Poincaré's recurrence theorem, $\mu(Y_n) = 0$ and, hence, $\mu(X \setminus \widetilde{X}) = 0$. □

1.9 Examples.

(i) **Liouville's Theorem.**

Let $U \subseteq \mathbb{R}^n$ be a bounded open set and let $f \in C^\infty(U, \mathbb{R}^n)$. Consider the autonomous ordinary differential equation

$$\dot{x} = f(x). \qquad (*)$$

By the standard existence and uniqueness theorem of ordinary differential equations, there exists, for any $p \in U$, a unique solution $\varphi_t(p)$ of $(*)$, with $\varphi_0(p) = p$. For simplicity, we assume that $\varphi_t(p)$ is defined for all $t \in \mathbb{R}$. We then have a one-parameter family of diffeomorphisms $\{\varphi_t\}_{t\in\mathbb{R}}$ which satisfies $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

We claim that the Lebesgue measure λ on U is invariant under the flow $\{\varphi_t\}_t$ if and only if $\operatorname{div} f = 0$, where $\operatorname{div} f = \partial_{x_1} f_1 + \cdots + \partial_{x_n} f_n$ is the divergence of the vector field $f = (f_1, \ldots, f_n)$.

Indeed, for an arbitrary compact subset A of $\mathbb{R}$ and for $h \in \mathbb{R}$, one has

$$\lambda(\varphi_{t+h}(A)) = \int_{\varphi_{t+h}(A)} dx = \int_{\varphi_t(A)} \left| \det \frac{\partial \varphi_h(x)}{\partial x} \right| dx.$$

A Taylor expansion around x shows that $\frac{\partial \varphi_h(x)}{\partial x} = \mathrm{id} + df(x)h + o(h)$, and hence

$$\det \frac{\partial \varphi_h(x)}{\partial x} = 1 + \operatorname{div} f(x)h + o(h).$$

Observe that $\det \frac{\partial \varphi_h(x)}{\partial x}$ is positive for h small enough. Hence,

$$\frac{d}{dt}\lambda(\varphi_t(A)) = \lim_{h\to 0} \int_{\varphi_t(A)} \frac{\operatorname{div} f(x)h + o(h)}{h} dx = \int_{\varphi_t(A)} \operatorname{div} f(x) dx.$$

Thus, if $\operatorname{div} f = 0$, then $\lambda(\varphi_t(A))$ is constant. By Poincaré's recurrence theorem

$$\liminf_{t\to\infty} d(\varphi_t(p), p) \leq \liminf_{n\to\infty} d(\varphi_n(p), p) = \liminf_{n\to\infty} d(\varphi_1^n(p), p) = 0$$

for almost all $p \in U$. The conclusion is that almost all orbits satisfy a recurrence property known as stability in the sense of Poisson.

(ii) Poincaré 's recurrence theorem and Liouville's theorem have the following paradoxical prediction. Consider a box containing some gas. Assume that at the initial time all gas molecules are concentrated in one half of the box. Then, after some time – in fact, infinitely often – the gas molecules will collect in the original portion of space. The resolution is this paradox is the following. The number of degrees of freedom is very large (under normal conditions, the number of gas particles in $1\,\mathrm{cm}^3$ is 10^{20}). Thus, the probability that the gas is in one half of the box behaves like $\lambda^{10^{20}}$, for some $\lambda < 1$, and the approximate period of the Poincaré recurrence is proportional to the inverse of this probability. It is clearly impossible to observe systems during such huge time intervals.

Topological properties such as compactness imply the existence of recurrent points.

1.10 Theorem (Birkhoff). *Let X be a compact space and $\varphi : X \to X$ be a continuous map. Then there is a recurrent point $x \in X$.*

Proof Consider the set $\mathcal{O}$ of all non-empty, closed and φ-invariant subsets of X. The finite intersection property shows that $\mathcal{O}$ is inductively ordered by inclusion. Hence, by Zorn's lemma, $\mathcal{O}$ contains a minimal element Y. Clearly, $Y = \overline{\{\varphi^n y \mid n \in \mathbb{N}\}}$ for all $y \in Y$, showing that every $y \in Y$ is recurrent. □

1.11 Examples.

(i) Let G be a *compact* group, $a \in G$ and $\varphi : G \to G$, $g \mapsto ag$. Then *all* points in G are recurrent. Indeed, by the previous theorem, there exists a recurrent point $x_0 \in G$. Let $g \in G$ be arbitrary and let $u = x_0^{-1} g$. If V is a neighbourhood of g, then $V \cdot u^{-1}$ is a neighbourhood of x_0. Hence, $a^n x_0 \in V \cdot u^{-1}$, that is, $a^n g \in V$ for some $n \in \mathbb{N}$.

(ii) Here is an application to a problem in diophantine approximation:

For any $\alpha \in \mathbb{R}$ and for any $\varepsilon > 0$, there exist integers n, m such that

$$|\alpha n^2 - m| < \varepsilon.$$

As to the proof, consider the two-dimensional torus

$$\mathbb{T}^2 = \{(\vartheta, \vartheta') \mid \vartheta, \vartheta' \in \mathbb{R}/\mathbb{Z}\}.$$

Define a continous mapping $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$\varphi(\vartheta, \vartheta') := (\vartheta + [\alpha], \vartheta' + 2\vartheta + [\alpha]),$$