London Mathematical Society Lecture Note Series. 257

An Introduction to Noncommutative Differential Geometry and its Physical Applications

Second Edition

John Madore
Université Paris XI

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge, CB2 2RU, UK www.cup.cam.ac.uk
40 West 20th Street, New York, NY 10011-4211, USA www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain

First published 1995
Second edition 1999
Reprinted with corrections 2000

A catalogue record for this book is available from the British Library

ISBN 0 521 65991 4 paperback

Transferred to digital printing 2002

Contents

1 Introduction

... Einstein was always rather hostile to quantum mechanics. How can one understand this? I think it is very easy to understand, because Einstein had been proceeding on different lines, lines of pure geometry. He had been developing geometrical theories and had achieved enormous success. It is only natural that he should think that further problems of physics should be solved by further development of geometrical ideas. How, to have $a \times b$ not equal to $b \times a$ is something that does not fit in very well with geometrical ideas; hence his hostility to it. *

If V is a set of points then the set of complex-valued functions on V is a commutative, associative algebra. As a simple example suppose that V has a finite number of elements. Then the algebra is of finite dimension as a vector space. The product of two vectors is given by the product of the components and it satisfies the inequality $\|fg\| \leq \|f\|\|g\|$ with respect to the norm $\|f\| = \max|f|$. Let f^* be the complex conjugate of f. Then obviously the product satisfies also the equality $\|ff^*\| = \|f\|^2$. A normed algebra with an *involution* $f \mapsto f^*$ which satisfies the above two conditions is called a *C^*-algebra.* Conversely any finite-dimensional commutative algebra which is a C^*-algebra can be considered as an algebra of functions on a finite set of points. The number of points is encoded as the dimension of the algebra. It is obviously essential that the algebra be commutative in order that it have an interpretation as an algebra of functions on a set of points. The finite-dimensional example has an interesting extension to infinite sets if they have a topology. If in fact V is a compact space then the subalgebra $C^0(V)$ of continuous functions on V is a C^*-algebra. It can be shown quite generally that conversely any commutative C^*-algebra with a unit element can be considered as an algebra of functions on a compact space. The space can be described using the language of statistical physics as the space of pure states of the algebra and when we pass to noncommutative geometry we shall see that a pure state is a natural generalization of the notion of a point.

If V is a smooth manifold then the algebra of smooth functions $\mathcal{C}(V)$ defined on it is of course a commutative algebra and in this case also there is an intrinsic characterization of the set of all such algebras, an additional structure which can be added to an arbitrary commutative algebra $\mathcal{A}$ which would insure that $\mathcal{A} = \mathcal{C}(V)$ for some V. The manifold V can always be considered as embedded in a euclidean space $\mathbb{R}^n$ of sufficiently high dimension. The coordinates of the embedding space are generators of an algebra of polynomials which is dense in the algebra $\mathcal{C}(\mathbb{R}^n)$ of smooth functions and the equations which define the manifold are relations in $\mathcal{C}(\mathbb{R}^n)$. The quotient of $\mathcal{C}(\mathbb{R}^n)$ by the ideal generated by the relations is equal to the algebra of smooth functions $\mathcal{C}(V)$ on V. Points of V can be again identified with pure states of $\mathcal{C}(V)$ and vector fields on V can be identified with the derivations of

*P.A.M. Dirac, as cited in *The Mathematical Intelligencer* **11** (1989) 58

$\mathcal{C}(V)$. It is however not possible in general to recover completely the differentiable and topological structure of V from the algebraic structure of $\mathcal{C}(V)$ alone. Only if the embedding functions are holomorphic or algebraic can the correspondence between the manifold and the algebra be one to one.

The aim of *noncommutative geometry* is to reformulate as much as possible the geometry of a manifold in terms of an algebra of functions defined on it and then to generalize the corresponding results of differential geometry to the case of a noncommutative algebra. We shall refer to this algebra as the *structure algebra.* The main notion which is lost when passing from the commutative to the noncommutative case is that of a point. 'Noncommutative geometry is pointless geometry.' The original algebra which inspired noncommutative geometry is that of the quantized phase space of non-relativistic quantum mechanics. In fact Dirac in his historic papers in 1926 (Dirac 1926a; Dirac 1926b) was aware of the possibility of describing phase-space physics in terms of the quantum analogue of the algebra of functions, which he called the quantum algebra, and using the quantum analogue of the classical derivations, which he called the quantum differentiations. And of course he was aware of the absence of localization, expressed by the Heisenberg uncertainty relation, as a central feature of these geometries. Inspired by work by von Neumann (1955) for several decades physicists studied quantum mechanics and quantum field theory as well as classical and quantum statistical physics giving prime importance to the algebra of observables and considering the state vector as a secondary derived object. This work has much in common with noncommutative geometry. The notion of a pure state replaces that of a point and derivations of the algebra replace vector fields. More recently Connes (1986) introduced an equivalent of the notion of an exterior derivative and generalized the de Rham cohomology of compact manifolds to the noncommutative case.

In Chapter 2 we shall give a brief review of ordinary differential geometry emphasizing those aspects which it is possible to generalize to the noncommutative case. We give as examples the 2-dimensional torus, the sphere and the pseudosphere, chosen because from them by a simple modification noncommutative geometries can be constructed. In Chapter 3 a noncommutative generalization of a metric and a linear connection are proposed. We give some examples of noncommutative geometries using the algebra M_n of $n \times n$ complex matrices as structure algebra. These examples have the advantage of being without any complications due to analysis. The Hodge part of the Hodge-de Rham theory is completely trivial whereas the de Rham part is still of some interest. Matrix geometry has also the advantage of being in some aspects identical to ordinary geometry and constitutes therefore a transition to the more abstract noncommutative geometries which are given in Chapter 4. In Chapter 5 a short review of the theory of vector bundles is given and their noncommutative generalizations are discussed. Most of the mathematical work in noncommutative geometry has been directed towards a generalization of the theory of differential operators on vector bundles over compact manifolds. In Chapter 6 cyclic cohomology is defined and two important results, Morita equivalence and the theorem of Loday and Quillen, are stated but not proven.

The last two chapters are devoted to the suggestion that noncommutative geometry might find interesting applications in high-energy physics. There has been renewed interest recently in the possibility that at very small length scales the structure of space-time is not properly described by a differentiable manifold. A natural alternative is offered by noncommutative geometry. In the conventional formulation of quantum field theory as the theory of formally quantized classical fields on a classical Minkowski space-time, ultraviolet divergences arise when one attempts to measure the amplitude of field oscillations at a precise given point in space-time. One way of circumventing this problem would be to give an additional structure to the point which would render impossible such measurements. For example, one could modify the microscopic structure of space-time with the hypothesis that at a sufficiently small fundamental length the coordinates of a point become noncommuting operators. This means in particular that it would be impossible to measure exactly the position of a particle since the three space coordinates could not be simultaneously diagonalized. We use here the language of quantum mechanics. An observable is an operator on a Hilbert space of states and the result of a measurement of an observable f when the system is in a state ψ of unit norm is given by $(\psi, f\psi)$. The act of measuring f forces the system into an eigenvector of f and so two observables can be simultaneously measured if and only if they commute. The position of a particle would no longer have a well defined meaning. Since we certainly wish this to be so at macroscopic scales, we must require that the fundamental length be not greater than a typical Compton wavelength. In other words, the fuzziness which the noncommutative structure gives a point in space-time could not be greater than the quantum uncertainty in the position of a particle. We can think of space as being divided into *Planck cells*, just as quantized phase space is divided into *Bohr cells*. The cellular structure replaces the point structure in the same way that the Bohr cells replace points in phase space when Planck's constant is not equal to zero. Gravity would in this approach appear as a deformation of the cellular structure and the graviton would become a sort of 'space-time phonon'. In Chapter 7 a geometry is described which is noncommutative at short length scales but which at large scales resembles the geometry of the ordinary 2-sphere. The geometry has the algebra M_n as structure algebra and there are n cells. The cellular structure is uniform as is the curvature. There is at present no consensus on what would be the most satisfactory noncommutative version of Minkowski space.

The question of whether or not space-time has 4 dimensions has been debated for many years. One of the first negative answers was given by Kaluza (1921) and Klein (1926) in their attempt to introduce extra dimensions in order to unify the gravitational field with electromagnetism. Einstein & Bergmann (1938) suggested that at sufficiently small scales what appears as a point will in fact be seen as a circle. Later, with the advent of more elaborate gauge fields, it was proposed that this *internal manifold* could be taken as a compact Lie group or even as a general compact manifold. The great disadvantage of these extra dimensions is that they introduce even more divergences in the quantum theory and lead to an infinite spectrum of new particles. In fact the structure is strongly redundant and most of it has to be discarded. An associated problem is that of localization. We cannot, and indeed do not wish to have to, address the question of the exact position of a particle in the extra dimensions any more than we wish to localize it too exactly in ordinary space-time. We shall take this as motivation for introducing in the last chapter a modification of Kaluza-Klein theory with an *internal structure* which is described by a noncommutative geometry and in which the notion of a point does not exist. As particular examples of such a geometry we shall choose only internal structures which give rise to a finite spectrum of particles.

Quite generally one can address the question of how far it is possible to transcribe all of space-time physics into the language of noncommutative geometry. We shall see in Chapter 6 that a differential calculus can be constructed over an arbitrary associative algebra. This would permit the formulation of gauge theories in any geometry. Matter fields could be incorporated as elements of algebra modules. In a less general setting a sort of Dirac operator has been proposed and a generalized integral (ConnesConnes 1992). This would permit the construction of an (euclidean) action. We shall discuss this briefly in Chapter 5. In this formulation space-time and the bosonic fields are incorporated in the algebra but the fermionic fields appear separately as modules. A more general unification in the spirit of supersymmetry would involve the elimination of the modules in favour of an algebra which is in some sense supersymmetric. Even in the simple matrix models which are discussed in Chapter 3 this has not been done in a completely satisfactory manner. A more serious problem is that of quantization. The 'Central Dogma' of field theory at the present time is that all information is contained in the classical action. The Standard Model is defined by a classical action which is assumed to contain implicitly all of high-energy physics. Quantum corrections are obtained by a standard quantization procedure. This quantization procedure has not been generalized to noncommutative models even in the simplest cases. The examples which have been used to propose classical actions which might be relevant in high-energy physics all involve simple matrix factors. They are quantized by first expanding the noncommutative fields in terms of ordinary space-time components and then quantizing the components. Under quantization the constraints on the model which come from the noncommutative geometry are lost.

Our purpose here is to furnish an elementary introduction to the subject of noncommutative geometry for non-specialists with special emphasis on the matrix case. Some knowledge of ordinary differential geometry including fibre bundles is supposed, although those parts of the subject are recalled which are useful for the understanding of the noncommutative case. We have assumed a few elementary notions and results from the theory of Hilbert and Banach spaces as well as from the theory of rings and algebras. Emphasis has been put on algebraic properties since our prime objects of interest are matrix algebras. This means that subtle, very often important, points of analysis are if at all but briefly mentioned. A certain familiarity with the rudiments of classical field theory has been also supposed since special emphasis has been placed on those aspects which might be useful in field theory and numerous examples have been taken from physics. Although the text is self-contained the examples given in some of the sections can include applications to subjects which require further knowledge.

Important words are placed in italics when first used and sometimes subsequently if the definition is extended. If the definition is not given in the text the word is assumed to be known and the definition can be found in the literature cited in the Notes. Quotes are sometimes used to underline the fact that a word or phrase is ill-defined. In particular, words which are normally only defined in commutative geometry and which are used intuitively in the noncommutative context are placed in quotes. We shall be primarily interested in the 'quasi-commutative' limit and we shall continue to use the word 'function', to designate an element of the algebra generated by the 'coordinates' even though the product of two 'functions' need not commute. We hesitate to use the words 'noncommutative manifold' or 'quantum manifold'. So 'noncommutative geometry' designates both the field of study and the object which is being studied.

We have tried to use conventional notation as far as possible and there is an occasional conflict between the conventions of physics and of algebra and geometry. The upper case H stands for homology, Hermite, Hamilton and Hall; the lower case e denotes an idempotent as well as the electron charge. The symbol δ designates functional variation as well as three different maps of cochains. In Chapter 2 a tilde is placed on quantities which will in subsequent chapters be replaced by noncommutative or 'quantized' equivalents. In quantum mechanics frequently physicists place a hat on an operator to distinguish it from the corresponding classical function. In this respect the tilde is an 'anti-hat'. The tilde is also used, especially in Chapter 8, to designate forms which have been lifted from a manifold to a bundle, as well as for related reasons. Unless otherwise indicated a tensor product is always over the complex numbers. If both factors are of infinite dimension then the symbol '$\otimes$' implies an appropriate completion with respect to the topology used on either factor. The symbol '$\mathcal{A}$' designates an associative algebra over the complex numbers with a unit element. If $\mathcal{A}$ is a formal algebra we shall often use the same symbol or name when referring to a topological completion as a C^*-algebra or a representation thereof as a von Neumann algebra. A commutator $[a, b] = ab \pm ba$ in a graded algebra will always be a graded commutator. This means that the minus sign is used unless the gradings of both a and b are odd. The symbol '$*$' is overworked. In front of a form it indicates the dual form; on a map it indicates an induced map and used as an index, as in H^* and H_*, it stands for a set of natural numbers. In this last usage the symbol '$2*$' stands for a set of even numbers and '$* - 1$' a shift by -1. Especially in Section 4.2 it indicates a product in an algebra; it is easier to place a hat on a star than on a point. We use the word 'Example' in a loose sense which indicates anything from a simple particular case to a subject of further research.

In the second edition some errors were corrected and a few more recent results were included, mostly concerning the definition of linear connections. A brief review of von Neumann algebras was also added since there have been some models proposed in which it might be possible to distinguish 'observationally' two representations of the same associative algebra. If one considers the algebra as the analogue of the theory and the representation as a choice of physical state this means that it will be possible to distinguish two states of the same theory. Some of the new material is based on articles written in collaboration with B.L. Cerchiai, Sunggoo Cho, M. Dubois-Violette, G. Fiore, R. Hinterding, Y. Georgelin, T. Masson, J. Mourad, K.S. Park, P. Schupp, S. Schraml, H. Steinacker and J. Wess.

In this new printing some errors were corrected and a few more recent results were included, all concerning the interface with string theory. Some of the new material is based on articles written in collaboration with S. Schraml, P. Schupp and J. Wess.

The author would like to thank A. Čap and M. Dubois-Violette for numerous helpful suggestions as well as F. Constantinescu, C. Duval, J. Gratus, H. Grosse, T. Masson, J. Mourad, F. Müller-Hoissen, A. Nicas, T. Schücker and F. Vanderseypen for pointing out several obscurities and errors in the original manuscript. While preparing the revised text we have benefited from conversations with B.L. Cerchiai, A. Chakrabarti, M. Durdević, G. Fiore, K. Fredenhagen, H. Grosse, A. Kehagias, C. Korthals-Altes, G. Landi, E. Langmann, M. Maceda, D. Madore, C. Ohn, O. Richter, K. Schmüdgen, T. Schücker, A. Turbiner, L. Vainerman, D. Vergara, J. Wess and C. Zachos. Much of the work was done while the author was visiting the Max-Planck-Institut für Physik in München. He would like to thank J. Wess also for his hospitality there.

Allein es steht in einem andern Buch,
Und ist ein wunderlich Kapitel. *

*Mephistopheles to Faust