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Singularity Theory

Proceedings of the European Singularities Conference, Liverpool, August 1996. Dedicated to C.T.C. Wall on the occasion of his 60th birthday.

Edited by

Bill Bruce
University of Liverpool

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University of Warwick

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Preface

Singularity Theory is a broad subject, involving a substantial number of mathematicians in most European countries. It was natural that a group should put together an application to the EU to set up a European Singularities Network (ESN) in 1993. It was equally natural that Terry Wall, of the University of Liverpool, should be selected to head up the bid and, subsequently, the organising committee.

One of the activities funded by the ESN was an international review meeting. Given that Terry Wall was born in 1936, a number of his friends, colleagues, and ex-students (some in all three categories) decided that, to honour him on his 60th birthday, the meeting should bear his name. It was held from the 18th to the 24th of August 1996, at the University of Liverpool. It is a sign of Terry's continued vigour in the subject that we felt it best to give a Web Page address (see the end of this preface) for a list of his current publications, rather than print that list at the time of going to press.

The meeting was attended by 88 mathematicians, 74 of them from 14 different countries outside the UK, and there were 61 talks. The festivities included a party hosted by Terry and his wife Sandra at their home, a multinational football (soccer) match, and a trip to the Lake District to experience a traditional English downpour.

The papers presented here are a selection of those submitted to the Editors for inclusion in the proceedings of the meeting. Pressure of space has meant that a substantial number of other high quality submissions could not be included. In the introduction we have given a brief review of the subject and have attempted to set the scene for the various contributions in this collection.

We would like to thank the EU for sponsoring the ESN, and consequently much of the expense associated to the meeting. We would also like to thank the London Mathematical Society for its financial help, which made it possible to include 10 mathematicians from the Moscow School of singularity theory in our invitations. We are grateful to Andrew Ranicki for coming and telling us of Terry Wall's exploits in his earlier life as a topologist, and to our tea-ladies Emily and Joan, for serving up the refreshments at the conference with such style. Finally many thanks to Wendy Orr for doing so much of the organisational work in her usual calm, friendly and efficient manner, and to Neil Kirk for his enormous help with a great deal of the technical editing.

Web Address

The first URL below refers to the Web pages for The Department of Mathematical Sciences at The University of Liverpool, England. Follow the link to the Pure Mathematics Research Division (this can be reached directly via the second URL if you prefer) and then follow the link to staff and postgraduates. (We decided not to give a direct link as these have the annoying habit of changing over time. The addresses below should be fairly stable!)

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University of Warwick

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Introduction

We start with a rapid survey, which we hope may be of some use to non-experts.

Singularity theory is a broad subject with vague boundaries. It is concerned with the geometry and topology of spaces (and maps) defined by C^∞, polynomial or analytic equations, which for one reason or another fail to be smooth, (or submersions/immersions in the case of maps). It draws on many (most?) other areas of mathematics, and in turn has contributed to many areas both within and outside mathematics, in particular differential and algebraic geometry, knot theory, differential equations, bifurcation theory, Hamiltonian mechanics, optics, robotics and computer vision.

It can be seen as a crossroads where a number of different subjects and projects meet. In order to classify its current diverse productions, we centre our discussion around the contribution of five major figures: Whitney, Thom, Milnor, Mather and Arnold.

The first of these to work on singularities was Whitney, who was led to study singularities in the process of proving his immersion theorem. An n-manifold M can be immersed in $(2n-1)$-space, even though immersions are not dense in the space of all maps $M \to \mathbb{R}^{2n-1}$: singularities persist under small deformations. To remove them one needs a large deformation, and a good understanding of the singularities themselves. Whitney wanted a short list of the singularities that persist; besides finding the list for maps $M^n \to \mathbb{R}^{2n-1}$ ([18]) he also did so for maps $M^2 \to \mathbb{R}^2$ ([19]). The persistence of certain singularities under deformation leads to the idea of a stable map: one which is essentially unchanged if we deform it a little. Two maps $M \to N$ are *right-left equivalent* if one can be converted into the other by composing with diffeomorphisms of source and target; a smooth map is *stable* if its equivalence class is open, in some reasonable topology, in $C^\infty(M, N)$. The weaker notion of topological stability replaces diffeomorphisms by homeomorphisms in source and target. Whitney characterised the stable mappings $M^n \to \mathbb{R}^{2n-1}$ and $M^2 \to \mathbb{R}^2$, and in doing so introduced many of the central ideas and techniques in the subject.

Thom was led to study singularities while considering the question of whether it is possible to represent homology classes in smooth manifolds by embedded submanifolds. With his transversality theorem ([16]) he gave the subject a push towards a kind of modern Platonism. Given smooth manifolds M and N, the jet bundle $J^k(M,N)$ is the space of Taylor polynomials of degree k of germs of smooth maps from M to N. It has a natural structure as a smooth bundle over $M \times N$, using the coefficients in the Taylor polynomials as fibre coordinates. If $f : M \to N$ is a smooth map, then by taking its k-th degree Taylor polynomial at each point, we obtain a map $j^k f : M \to J^k(M,N)$, the jet extension map. Thom saw the jet bundle as a version of the Platonic world of disembodied ideas, partitioned into attributes (the orbits of the various groups which act naturally on jets) as yet unattached to the objects (functions and mappings) which embody them. A map $f : M \to N$ embodies an attribute W faithfully if its jet extension map $j^k f : M \to J^k(M,N)$ is transverse to W. Thom showed that for all M, N and W, most maps $M \to N$ (actually a residual set of maps) have this property. This theorem put the "generic" into every subsequent theorem about generic behaviour of maps, in particular Mather's theorems that stable maps $M \to N$ are dense in $C^\infty(M,N)$ if $(\dim M, \dim N)$ are in his range of nice dimensions (Mather [8]–[13]), and that topologically stable maps are dense, in all dimensions [14]. It does the same for the applications of singularity theory to differential geometry, a field which is represented in this collection by the papers of Ishikawa and Donelan-Gibson.

Thom also contributed the idea of a versal unfolding. Closely related to the idea of a slice to a group orbit, a versal unfolding of a map or map-germ f is a finite-dimensional family $f_\lambda, \lambda \in \Lambda$ which explores every nearby possibility of deformation, up to some specified notion of equivalence. Versal unfoldings of minimal dimension are unique up to isomorphism, and the structure of its miniversal unfolding provides one of the best ways of understanding a singularity. A germ $f : (M,x) \to (N,y)$ has a versal unfolding with respect to a given equivalence relation if it has "finite codimension"; that is, if its orbit in the jet bundle $J^k(M,N)$, with respect to this equivalence relation, has codimension which is eventually independent of k. The term "versal" is the intersection of "universal" and "transversal", and one of Thom's insights was that the singularities of members of families of functions or mappings are versally unfolded if the corresponding family of jet extension maps is transverse to their orbits (equivalence classes) in jet space.

This insight, and Thom's Platonist leanings, led him to Catastrophe Theory. He identified and described the seven orbits of function singularities which can be met transversely in families of four or fewer parameters: these were his seven elementary catastrophes, which were meant to underly all abrupt changes (bifurcations) in generic four-parameter families of gradient dynamical systems. With the eye of faith some of his followers were able to see the elementary catastrophes in every field of science; without it, their critics objected to the invocation that this entailed of such invisibles as real variables and smooth potential functions in fields like politics and prison riots, and to the triteness of some of its conclusions. Nevertheless many of Thom's ideas in bifurcation theory and gradient dynamical systems have provided the basis for later development, and the controversy surrounding

Catastrophe Theory should not mask the importance of his contribution to the subject.

Besides proving Thom's conjectures about the genericity of stability, Mather made possible the systematic classification of germs of functions and maps. There are several standard equivalence relations, induced by the action of the various groups of diffeomorphisms on the space of map-germs. Mather showed that if a germ $f : (M, x) \to (N, y)$ has finite codimension with respect to a given equivalence relation, then it is *finitely determined* with respect to that relation: there is an integer k such that any other germ having the same k-jet as f, is equivalent to it. Subsequently other authors, beginning with Jean Martinet, Terry Gaffney and Andrew du Plessis, improved on Mather's estimates for the determinacy degree (the smallest such k); see the excellent survey article of Wall [17]. Their work led to a number of papers providing lists of polynomial normal forms for equivalence classes of low codimension. Terry Wall is also a prodigious list-maker — see for example [105],[110],[129] in his list of publications. Indeed, the presence of a large taxonomy is one of the features of contemporary singularity theory, and significant theoretical advances have often come from the desire to understand features empirically observed in the lists. The paper of Houston and Kirk in this collection uses recently developed techniques from classification theory to obtain a new list, namely a list of right-left equivalence classes of singularities of maps from 3-space to 4-space.

One of the most influential mathematicians working in singularity theory is V.I. Arnold, also a prodigious list-maker. In the early seventies, in connection with a project to extend the stationary phase method for the estimation of oscillatory integrals in quantum mechanics, Arnold produced extensive lists of singularities of functions ([1]–[4]) making spectacular advances in the techniques of classification, and in understanding the structure of the objects and the lists he found. He also introduced his notions of Lagrangian and Legendrian singularities, which systematise many of the mathematical ideas behind Catastrophe Theory. The reader is recommended Arnold's little book [5], both for a lively, personal, and probably overly harsh view of Catastrophe Theory and, more interestingly, for its exposition of the important contributions made by his school to a number of interesting geometrical problems, using techniques from singularity theory.

One of Arnold's contributions was the crucial concept of *simple singularity* — one whose classification does not involve continuous invariants — which has proved extremely fruitful. One of the oddities of a subject with many lists is that the same lists may occur with different headings, and none more so than the short list of simple singularities determined by Arnold. The singularities in Arnold's list had already appeared as Kleinian singularities, and as the "singularities which do not affect adjunction" in a paper of Du Val in 1934. In [6], Durfee gives 15 different characterisations. Several of them, due to Brieskorn, come from the theory of algebraic groups. Arnold also noticed a mysterious duality between some of his lists, which was subsequently interpreted by a number of mathematicians and extended by Terry Wall and Wolfgang Ebeling. Ebeling's paper in this collection surveys this and further extensions of Arnold's strange duality, and relates it to mirror symmetry.

Singularity theory typically focuses on local behaviour, on germs of spaces and maps. Isolated singular points of algebraic and analytic varieties are studied by considering only what goes on inside a conical neighbourhood, small enough to exclude the global topology and geometry of the variety. One then looks not only at the singular space, or map, but also at what goes on within this neighbourhood as the singularity is perturbed. Building on earlier work of Burau, Zariski, Mumford and Brieskorn, Milnor developed this approach in his 1968 book [15] (see also [7]), showing that if the zero set of a polynomial $f : \mathbb{C}^n \to \mathbb{C}$ has an isolated singular point, then the intersection of a nearby non-singular level set of f with a small ball B_ε around the singular point is homotopically equivalent to the wedge of a finite number of n-dimensional spheres. The number of these spheres reflects the complexity of the singularity; it is known as the Milnor number, μ, and the intersection itself is the Milnor fibre. In fact the restriction of f to B_ε defines a fibration, the Milnor fibration, over a small punctured disc around the critical value. A loop around zero in this disc lifts to a diffeomorphism of the Milnor fibre, inducing an automorphism of its homology. The determination of the monodromy automorphism of the singularities in Arnold's lists was one of the great achievements of his school. The study of the topology and algebraic geometry of isolated singular points of complex hypersurfaces remains the focus of a great deal of activity. It has developed a formidable technical armoury, crowned by Deligne's mixed Hodge theory, which places canonical filtrations on the cohomology of the Milnor fibre. The application of mixed Hodge theory to singularity theory was developed independently in the mid seventies by Joseph Steenbrink and Alexander Varchenko. Steenbrink's paper in these proceedings is concerned with his extension of this work to the construction of a mixed Hodge structure on the cohomology of the Milnor fibre of an isolated complete intersection singularity.

The idea of looking at what goes on in the neighbourhood of a singular point generalises naturally to other contexts in singularity theory: one should look for the nearby stable object. For an unstable map-germ f, the analogue of the Milnor fibre is therefore a *stable perturbation*; for a non-generic hyperplane arrangement, one sees what happens when it is moved into general position. Both cases are treated in papers in this volume: Marar-Montaldi-Ruas and Fukui compute the number of stable singularities of given type appearing in stable perturbations of certain unstable map-germs, using commutative algebra techniques centred on the properties of Cohen-Macaulay rings, and Damon proves a conjecture of Varchenko on multi-valued functions on the complements of affine hyperplane arrangements, by viewing the arrangements as "singular Milnor fibres" (nearby objects, stable in an appropriate class of spaces) of certain singularities.

Singularity theory is also concerned with the global behaviour of singular spaces. Singular algebraic and analytic spaces can be *stratified*, that is, given a locally finite partition into manifolds. One of Whitney's most fruitful contributions was to give geometric conditions (which now bear his name) for the local triviality (product-like structure) of a stratification. The study of stratified spaces led to the development, by Goresky and MacPherson, of Intersection Cohomology, in which a version of Poincaré duality for singular spaces is recovered, and to the introduction of the category of perverse sheaves. Whitney stratifications were also the key to Mather's proof of the topological stability theorem, and to work of many authors on conditions for topological triviality of families of functions and maps. Recent developments are surveyed here in the paper of Terry Gaffney and David Massey. Two papers on stratifications, by Andrew du Plessis, provide the first correct proofs of key results in the area of real stratification theory, and the paper of Brasselet and Legrand looks at differential forms on stratified spaces and describes a version of Connes's celebrated theorem in which the de Rham cohomology of a smooth manifold is computed from the cyclic homology of its algebra of smooth functions.

Singular sets arise naturally as the images and discriminants of smooth mappings. The paper of Kevin Houston presented here describes a spectral sequence for computing the homology of such sets from the alternating homology of the multiple point sets of the map.

One of the areas of greatest interest over the last few years has concerned the jacobian conjecture: that a polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ with nowhere-vanishing (and therefore constant) jacobian determinant, is an isomorphism. Attempts by Lê and Weber to prove this conjecture by looking at the behaviour of polynomials at infinity have led to a number of papers on this topic, represented here by the contribution of Mihai Tibar.

We hope we have convinced you that singularity theory is indeed a broad subject. This brief introduction can only hope to cover a part of the large area of mathematics it encompasses. Indeed, the topics represented by the articles in this volume, some of which are mentioned above, give a good indication of the diversity of the subject. The following section provides short summaries of each paper, which we hope the reader will find useful.

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Summaries of the Papers

Complex Singularities

Singularities Arising from Lattice Polytopes
K. Altmann

Toric geometry begins with a procedure for assigning to each integral polyhedral cone σ in $\mathbb{R}^n$ an algebraic variety V_σ, and with each inclusion of one cone in another a morphism of the corresponding varieties. The varieties obtained in this way are called *toric varieties*, and include affine space, projective space, and many other important algebraic varieties. Since its introduction twenty years ago, toric geometry has become an important tool for algebraic geometers; the combinatorics of lattice polytopes encodes important information about the algebraic geometry of the toric varieties they give rise to. Klaus Altmann's paper begins with a brief introduction to toric geometry and goes on to survey some of the links between the geometry of polytopes and the singularities of toric varieties, including their deformation theory.

Critical Points of Affine Multiforms on the Complements of Arrangments
J.N. Damon

A hypersurface $(D, 0)$ in a complex manifold $(X, 0)$ is a *free divisor* if the sheaf $\mathrm{Der}(\log D)$ of vector fields on X tangent to D is free as an $\mathcal{O}_X$-module. Free divisors occur throughout singularity theory, as discriminants of stable maps and of versal deformations of singularities. If $\phi : (Y, y) \to (X, 0)$ is transverse to the free divisor D outside y then $E = \phi^{-1}(D)$ is an *almost free divisor*, and if ϕ_t is a deformation of ϕ which is everywhere transverse to D, then $E_t = \phi_t^{-1}(D)$ is a *singular Milnor fibre* of E. The freeness of D allows one to calculate the rank of the vanishing homology of E_t as the length of a certain module defined in terms of ϕ and $\mathrm{Der}(\log D)$, and Damon has made spectacular use of this fact in a number of papers. Here he further develops these ideas to prove a strengthened version of a conjecture of Varchenko on critical points of multivalued functions defined on the complement of a hyperplane arrangement. Varchenko's conjecture originated in his work on the KZ equation described in this volume in the paper of Looijenga.

Strange Duality, Mirror Symmetry and the Leech Lattice
W. Ebeling

This paper surveys a series of curious dualities, originally concerning Arnold's list of 14 exceptional unimodal (or "triangle") singularities, and later extended by the author and Terry Wall to cover a wider class. The author describes interpretations of this strange duality, due to Pinkham, and, independently, to Dolgachev and Nikulin, and shows that these too can be extended to cover the Ebeling-Wall version. He also describes the connection, observed by M. Kobayashi, between Arnold's strange duality and mirror symmetry, and the relation found by K. Saito with the Leech lattice, famous as a hunting ground for sporadic simple groups.

Geometry of Equisingular Families of Curves
G.-M. Greuel and E. Shustin
The simplest, and for many the most familiar singularities, are those presented by plane curves. Their local topology was first investigated in the late 1920's, and they have provided an excellent proving ground for theories and theorems ever since. This paper addresses three central problems concerning plane curve singularities. Suppose we are given a natural number d and a finite list of types of plane curve singularities. We can then ask:
(1) Is there an (irreducible) curve of degree d in the complex projective plane $\mathbb{CP}^2$ whose singularities are precisely those in the given list?
(2) Is the family of such curves smooth, and of the expected dimension?
(3) Is this set connected?

These questions have a long history, appearing in the works of Plücker, Severi, Segre and Zariski. They are part of a more general problem: that of moving from local information to global results. The authors have made spectacular recent progress, which is surveyed in this paper. They extend their discussion to curves on more general surfaces. Combined with recent ideas of Viro their results also have application in real algebraic geometry.

Arrangements, KZ Systems and Lie Algebra Homology
E.J.N. Looijenga
The Knizhnik-Zamalodchikov differential equation first appeared in theoretical physics in connection with models for quantum field theory. A rather difficult paper of Varchenko and Schechtman established connections between this equation and the cohomology of local systems on the complements of hyperplane arrangements: under certain circumstances a complete set of solutions to the KZ equation is provided by generalised hypergeometric integrals associated to the hyperplane arangement. Looijenga's paper gives a largely self-contained account of the work of Varchenko and Schechtman and other related work, in the process providing considerable clarification of some of the main ideas. In particular it contains a clear and concise sheaf-theoretic treatment of the cohomology of the complement of a hyperplane arrangement.

The Signature of $f(x,y)+z^N$
A. NEMETHI
If $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ and $g:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ have isolated singularity then so does the sum $f+g:(\mathbb{C}^{m+n},0)\to(\mathbb{C},0)$. Classical theorems of Thom and Sebastiani describe the Milnor fibre F_{f+g} of $f+g$ and its monodromy automorphism in terms of the Milnor fibres and monodromy automorphisms of f and g. In case $m=2$ and $n=1$, F_{f+g} is a 4-manifold, so that the intersection pairing on its middle dimensional homology is symmetric, and thus a quadratic form. One of the most important invariants of a simply connected 4-manifold is the signature of this quadratic form; it is often referred to simply as the signature of the manifold. In this paper Andras Nemethi collects together results, many of them his own, which compute the signature of F_{f+g} in terms of invariants of f and g.

The Spectra of $\mathcal{K}$*-Unimodal Isolated Singularities of Complete Intersections*
J.M. STEENBRINK
Steenbrink and W. Ebeling recently extended Steenbrink's earlier construction of a mixed Hodge structure (MHS) on the Milnor fibre of a hypersurface singularity: an MHS can now be defined on the Milnor fibre of any isolated complete intersection singularity (ICIS) (X,x). The construction involves embedding (X,x) as a hypersurface in an ICIS (X',x) of minimal Milnor number. In fact, as in the case of hypersurface singularities, the MHS itself is very hard to calculate explicitly, and the closest one can get to it is the Hodge numbers (the dimension of the graded pieces of the associated bigraded structure) and the *spectrum* of the singularity, which is defined in terms of the monodromy action on the MHS. The spectrum is a collection of rational numbers, symmetric about its midpoint. In a certain precise sense it is semi-continuous under deformation, and indeed constant under μ-constant deformation. This semi-continuity is, together with its calculability, the key to its importance: it can be used to show that certain adjacencies of singularities do not occur.

In this paper Steenbrink calculates the spectrum for the unimodal ICIS's of embedding codimension at least 2, the calculation for hypersurface singularities having been made by Goryunov some years ago. The modality of a singularity is the number of continuous parameters involved in its classification; thus, "simple" = 0-modal. The unimodal hypersurface singularities were classified by Arnold in 1973, and the remainder of the list of unimodal ICIS's was determined in 1983 by Terry Wall (paper 105 in the list of his publications).

Dynkin Graphs, Gabriélov Graphs and Triangle Singularities
T. URABE
It is of interest to describe not only the topology of the Milnor fibre of a singularity (X,x), but also the constellations of singularities which can appear on singular fibres in a deformation of (X,x). In this paper the author gives a

simple algorithm for determining which constellations of simple singularities can appear on the fibres of a deformation of a triangle singularity. The 14 triangle singularities are a class of unimodal hypersurface singularity; under the name "exceptional unimodal singularities" they are the subject of the strange duality reported on by Ebeling in his paper in this volume. The name "triangle singularities" arose because they can be realised as quotients of $\mathbb{C}^2$ by certain finite groups of automorphism, constructed from triangles in the hyperbolic plane. The algorithm is in terms of the combinatorics of Dynkin diagrams (weighted graphs which encode information about the intersection form on the Milnor fibre).

Stratifications and Equisingularity Theory

Differential Forms on Singular Varieties and Cyclic Homology

J.-P. BRASSELET AND Y. LEGRAND

Brasselet has worked for a number of years on the global topology of stratified sets. In collaboration with Goresky and MacPherson, the creators of intersection homology, he has proved a version of the de Rham theorem for simplicial spaces. Another special feature of manifolds is provided by the de Rham theorem which (roughly) asserts that the cohomology of the complex of differential forms and exterior derivatives yields (over the reals) a ring isomorphic to the singular cohomology ring. The introduction of intersection homology theory led naturally to a search for a result analagous to de Rham's for a Whitney stratified set. Clearly what is required is some extension of the notion of smooth differential forms to stratified spaces.

This paper is inspired by the celebrated theorem of Alain Connes, which recovers the de Rham cohomology of a smooth manifold by means of the cyclic homology of the Fréchet algebra of smooth functions on the manifold. Brasselet and Legrand prove a version of this result for Whitney stratified spaces, in the process introducing a new definition of differential form on a stratified space. The key is the definition of a Fréchet algebra of functions on the regular part of the stratification. In this paper the authors associate to these objects a complex of differential forms and a Hochschild complex of forms on the regular part, with poles along the singular part. They then relate the de Rham cohomology of the first part, and the cyclic cohomology of the second, with the intersection homology of the stratified set.

The next two papers are motivated by questions emerging from the proof of the topological stability theorem.

Continuous Controlled Vector fields

A.A. DU PLESSIS

Frequently smooth equivalence of map-germs is too fine; it preserves too much detail, distinguishing phenomena one wished it did not. One then has to replace diffeomorphism by homeomorphism. Whereas diffeomorphisms are produced by integrating smooth vector fields, to produce the homeomorphisms needed to prove topological stability, for example, one has to relax the smoothness condition on the vector fields while retaining enough control to ensure that one can integrate and obtain continuous flows. This is done by stratifying the space and constructing the vector fields on the individual strata. To ensure that the flows on the strata fit together to yield a family of homeomorphisms, the stratification has to satisfy Whitney's regularity conditions, and the vector fields themselves must satisfy certain technical conditions (be "controlled").

In this paper the author looks at the following important special case. Suppose we are given a stratified manifold and a projection to a space which is a submersion on strata. This is the situation which arises when trying to prove that a given stratification is a topological product. If the stratification is Whitney regular (or indeed satisfies a weaker condition known as C- regularity, due to K. Bekka) then one can lift a smooth vector field in the target to one in the source which is controlled, hence integrate it and obtain the required trivialisation. Here du Plessis shows that one can also ensure that the lifted field is actually continuous.

Finiteness of Mather's Canonical Stratification

A.A. du Plessis

One of the triumphs of singularity theory is Mather's theorem that topologically stable maps are dense. In a series of papers Mather showed how to construct a "canonical" Whitney-regular stratification of each jet space, with the property that a proper map whose jet-extension is transverse to the stratification is topologically stable. In particular such mappings are open and dense in the space of all proper mappings.

Mather also claimed that this stratification has only finitely many connected strata. This is an important fact, since it implies that there are only finitely many topological types of topologically stable map-germs for any given source and target dimension. This paper gives the first complete proof of this fact. It also gives a new simpler construction of the canonical stratification, which is of substantial independent interest.

Trends in Equisingularity Theory

T.J. Gaffney and D. Massey

In many situations classification up to smooth equivalence yields continuous families of inequivalent types (or moduli). Finding coarser relations which yield finite classifications is the main aim of equisingularity theory. There are a number of different approaches: the purely topological, developed first by Lê and Ramanujam, using Smale's h-cobordism theorem; the stratification-theoretic, which looks for invariants whose constancy should guarantee Whitney regularity, developed initially by Teissier; and a third approach using filtrations on the spaces of map-germs, due mainly to Damon. In this paper the authors survey work on the second approach, discussing extensions (for which they are responsible) of the early work of Teissier, to cover the case of families of complete intersection singularities, and of non-isolated hypersurface singularities.

Regularity at Infinity of Real and Complex Polynomial Functions
M. Tibar
Let $f : K^n \to K$ be a polynomial mapping, with K the real or complex numbers. Classical singularity theory gives us a good understanding of the local topology of the fibres of f, but the global topology still presents a challenge. A value $a \in K$ is called typical if the map f is a local fibration near a. There are only finitely many atypical values. Some are critical values, where the mapping fails to be a fibration for obvious reasons, and the local results alluded to above explain how the fibres of f change near a. The other atypical points arise, roughly speaking, because of problems "at infinity". To obtain control of the mapping at infinity the author uses two regularity conditions: t- and ρ-regularity. The first hinges on a compactification of the map f, the second seeks a control function in the affine space which defines a codimension one foliation which can be used to control lifts of a constant (non-zero) vector field in K. The paper investigates the relationships between these triviality conditions and affine polar curves, over both $\mathbb{R}$ and $\mathbb{C}$.

Global Singularity Theory

A Bennequin Number Estimate for Transverse Knots
V.V. Goryunov and J.W. Hill
Vassiliev's approach to knot invariants, motivated by ideas from singularity theory, has proved astonishingly successful. The paper presented uses this approach to consider Legendrian knots in the standard contact 3-space $\mathbb{R}^3$. It is well known that any topological knot type in a contact 3-manifold has a Legendrian representative, that is, there is a curve of the same knot type whose tangent lines are contained at each point in the associated contact plane. When the contact structure is coorientable any Legendrian knot inherits a natural framing. A result of Bennequin shows that the self-linking (or Bennequin) numbers of all canonically framed Legendrian representatives of a fixed knot are bounded from one side. A new proof is presented here of the fact that these Bennequin numbers are bounded from below by the negative of the lowest degree of the framing variable in its HOMFLY polynomial. For knots with at most 8 double points in their diagrams these two numbers coincide.

Elimination of Singularities: Thom Polynomials and Beyond
O. Saeki and K. Sakuma
The theory of elimination of singularities is concerned with the following global problem: given a smooth map $f : M \to N$, does there exist a smooth map g homotopic to f which has no singularities of a prescribed type Σ? The

first, and most basic, obstruction to the existence of such a map g comes from the Thom polynomial, a polynomial in the Stiefel-Whitney classes of M and N which is dual to the homology class represented by the closure of the set of points of type Σ, since this only depends on the homotopy type of f. This article discusses circumstances in which this is the only obstruction.

Abelian Branched Covers of the Projective Plane
A. LIBGOBER
It is a classical problem, first studied by Zariski and van Kampen, to understand how the type and relative position of the singularities of a plane projective curve are reflected in the fundamental group of its complement. In this paper the author outlines a relationship between the fundamental group of the complement of a reducible plane curve, and certain geometric invariants depending on the local type and configuration of the singularities on the curve.

Singularities of Mappings

Multiplicities of Zero-Schemes in Quasihomogeneous Corank-1 Singularities
W.L. MARAR, J.A. MONTALDI AND M.A.S. RUAS

Butterflies and Umbilics of Stable Perturbations of Analytic Map-Germs
T. FUKUI
In the nice dimensions, every map-germ f of finite codimension can be perturbed so that it becomes stable. Since the bifurcation set is a proper analytic subset of the (smooth) base of a versal unfolding, any two stable perturbations are smoothly equivalent, and thus there is a well-defined stable perturbation f_t of f. Understanding it is an important problem, both for theoretical reasons, and for applications to other subjects. Besides the homology of its image, there are other integer invariants of importance, such as the number of nodes on a stable perturbation of a germ of a parametrised plane curve, or the number of critical points of a stable perturbation of a function-germ. These two papers give formulae for related 0-dimensional invariants of weighted homogenenous map-germs, in terms of their weights and degrees. Both use the principal of conservation of multiplicity for Cohen-Macaulay modules.

Fukui's paper considers map-germs $f : (\mathbb{C}^5, 0) \to (\mathbb{C}^4, 0)$. Here there are two local singularity types occuring at isolated points in a stable perturbation, namely A_4 and D_4. Marar, Montaldi and Ruas look at corank-1 map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and compute all 0-dimensional invariants, including those involving multi-germs.

An Introduction to the Image-Computing Spectral Sequence
K.A. HOUSTON
The images of analytic maps of complex manifolds are generally highly singular, even when the maps are stable: parametrised plane curves have nodes,

and surfaces mapped stably into 3-space have normal crossings and Whitney umbrellas. On the other hand, Sard's theorem assures us that the generic fibres of a map are smooth. Consequently, in order to understand the topology of a space which is the image of a finite map between smooth manifolds, one has to use methods which take account of this fact. A useful handle on such questions is provided by the *multiple point spaces* of the map: the closure of the set of k-tuples of pairwise distinct points with the same image. The symmetric group S_k acts naturally on the k-th multiple point space, permuting the points. This paper surveys work on a spectral sequence which relates the homology of the image to the alternating part of the homology of the multiple point spaces, and discusses applications to the computation of the homotopy type of generic images.

On the Classification and Geometry of Corank-1 Map-Germs from Three-Space to Four-Space

K.A. Houston and N.P. Kirk

Classifying map-germs up to right-left equivalence is far from easy, and rather few lists of equivalence classes have been obtained. The tables in this paper summarise a new classification and also show geometrical information about the germs listed. The key geometrical invariant of a germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is its *image Milnor number* $\mu_I(f)$, the rank of the middle homology of the image of a stable perturbation. It is conjectured that the image Milnor number and the $\mathcal{A}_e$-codimension of a map-germ satisfy the Milnor-Tjurina relation: $\mu_I(f) \geq \mathcal{A}_e\text{-codim}(f)$ with equality if f is weighted homogeneous, in analogy with the relation between Milnor number and Tjurina number of an isolated complete intersection singularity. This has been proved only when $n = 1$ and $n = 2$. The examples here provide the first support for the conjecture outside the range of dimensions where it has been proved. The paper also provides a brief survey of current classification techniques, and of results on the topology of map-germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ for $n < p$.

Applications of Singularity Theory

Singularities of Developable Surfaces

G. Ishikawa

Differential geometry, in particular the differential geometry of curves and surfaces in 3-space, has provided an exciting area of application for singularity theory. Initially it seems slightly surprising that singularity theory should have anything new to say about such a well-studied area. In fact its success here has been quite dramatic.

It has taken place on three fronts: the understanding of higher order phenomena (classical differential geometry concerns itself largely with quadratic forms); the understanding of generic phenomena, via transversality and classification theorems (classical geometry illustrates itself with a surprisingly small collection of often very special surfaces); and the understanding of the singular objects which arise naturally in connection with smooth embeddings (such as the developable surface of a curve, and the dual and focal sets of a surface).

In this paper the author surveys results on developable surfaces, those whose Gaussian curvature vanishes identically. It is a classical result that there are three types of analytic developable surface: cones, cylinders, and tangent developables (the union of the lines tangent to a space curve). In a natural sense the most general and interesting of these are the tangent developables. They are highly singular along the curve itself, and classical differential geometers found these singularities a source of embarrassment. In singularity theory they are, however, an object of great interest. After a brief discussion of some of the classical ideas using projective duality, Ishikawa describes the smooth and topological classification of the singularities of these surfaces.

Singular Phenomena in Kinematics

P.S. Donelan and C.G. Gibson

The study of the kinematics of rigid bodies received substantial impetus during the industrial revolution, and has more recently received new stimulus from robotics. Classical results such as the solution by Watt and Peaucellier of the problem of the conversion of circular to rectilinear motion, and Kemp's theorem that any plane algebraic curve can be traced by a mechanism, lie on a fascinating interface between geometry and engineering. The advances of kinematics in the 19th century can, in part, be traced to developments in the algebraic geometry of plane curves.

The thesis set out here is that the generic phenomena of rigid body kinematics can be understood using singularity theory. The basic object under review can be described as follows. Suppose given a mechanism of some form, for simplicity in the plane, with r degrees of freedom. If we attach another copy of the plane to the mechanism we will obtain an r-parameter family of motions of the plane, and in particular for each point of the plane a map from an r-dimensional space to the plane. If $r = 1$ this gives a 2 parameter family of curves. Most are immersed with transverse intersections, but for certain special tracing points we must expect, for example, cusps. Such points will be of some engineering interest. Understanding, in a general context, the types of singularity that can occur, and the nature of the corresponding exceptional tracing points is part of the task set here. The paper gives a detailed survey of the results obtained to date in this area.

Singularities of Solutions of First Order Partial Differential Equations

S. Izumiya

Singularity theory has applications to any area whose fundamental tools involve the calculus, and there is now a substantial body of work on its applications to bifurcation problems in ordinary differential equations. This part of the subject now has an independent existence, rather separate from singularity theory itself.

Izumiya's paper surveys some applications of singularity theory to partial differential equations. He confines himself to Hamilton-Jacobi equations and single conservation laws. Both equation types arise in a number of applications (eg. to the calculus of variations, differential games, and gas dynamics). Classically, solutions are obtained by the method of characteristics; initially such characteristics are smooth, but beyond a certain critical time they become multivalued. The classification of the associated singularities is discussed, as is the evolution of shock waves, in the two variable case.