Singularities Arising from Lattice Polytopes

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1 Introduction

(1.1) Assume that an affine variety $Y \subseteq \mathbb{C}^w$ is defined by certain binomials $\underline{z}^a - \underline{z}^b$ ($a, b \in \mathbb{N}^w$); for example take $Y := [z^n - xy] \subseteq \mathbb{C}^3$. Then, the ring of regular functions on Y equals the semigroup algebra $\mathbb{C}[S]$ with S obtained from $\mathbb{N}^w$ via identifying a and b. If, moreover, the semigroup S is easy to handle (for instance, if S is the set of lattice points in a rational, polyhedral, convex cone in some $\mathbb{R}^k$), then one might hope that important features of the algebraic variety Y can be encoded with combinatorial data.
This is, more or less, the main idea of the concept of (affine) toric varieties. A lot more has been done: if a bunch of polyhedral cones comes in a so-called fan, then the affine toric varieties associated to these cones glue together; if the fan arises from the inner normals of a lattice polytope, then we obtain a projective variety. These ideas have been developed over the last 20 years, and many textbooks are now available. For a detailed treatment we refer to [Da], [Fu], [Ke], or [Od]. For a short introduction to the subject without proofs see §2.

(1.2) As just mentioned, lattice polyhedra are related to projective toric varieties. Hence it is no surprise that (affine) cones over projective varieties, in the algebro-geometric sense, arise from cones over lattice polytopes in the sense of convex geometry.
Doing toric geometry, one has to deal with cones and their duals as well (cf. (2.1)). This implies that lattice polytopes have a second chance to induce a certain class of affine toric varieties; it was first observed by Ishida in [Ish], that this class consists exactly of the affine toric Gorenstein varieties. A more detailed explanation of both methods to construct singularities from lattice polytopes is given in (2.6) and (2.7).

(1.3) The main purpose of the present paper is to give a survey of the deformation theory of toric singularities (or equivalently, of affine toric varieties) known so far. In the very beginning, deformation theory appeared as the investigation of how complex structures may vary on a fixed compact, smooth manifold (cf. [Kod]). In a similar manner, we may regard deformations of germs of analytic spaces. If $Y = (Y, 0)$ is such a germ (often called "singularity", since smooth germs are not the interesting ones), we define the following functor:

$$\mathrm{Def}_Y((T,0)) \;:=\; \{\text{isomorphism classes of flat } g : Z \to T\,, \text{ together with } g^{-1}(0) \xrightarrow{\sim} Y\}\,.$$

Good references for facts about deformation theory of germs are Artin's Lecture notes [Art], the long introduction to Palamodov's paper [Pa], or Stevens' Habilitationsschrift [St 2]. In many cases, for instance for isolated singularities, there exists the so-called mini-versal deformation. By definition, it yields every possible deformation via specialization of parameters, i.e. via base change. The mini-versal deformation is, up to non-canonical isomorphism, uniquely determined and may be considered a source of numerical invariants of the original singularity. If Y is a complete intersection, then every perturbation of the defining equations yields a (flat) deformation of Y; in particular, its mini-versal deformation is a family over a smooth base space with well-known dimension.
However, as soon as we leave this class of singularities, the structure of the versal family or even the base space will be more complicated. The first example showing that the situation is not always as boring as in the complete intersection case was given by Pinkham, cf. [Pi]. He studied the cone over the rational normal curve of degree four; here the versal base space consists of two irreducible components of dimension one and three, respectively. In the following we will see similar examples in higher dimensions; even non-reduced points will occur as versal base spaces.

(1.4) What are the major issues we are concerned with? In the sequel we will discuss the following three:

(1) Study the vector spaces T^1_Y of infinitesimal deformations and T^2_Y containing the obstructions for lifting deformations onto larger base spaces. There are different ways of defining them (cf. (3.1)-(3.3)); but in any case, they are as multigraded as the semigroup algebra $\mathbb{C}[S]$ is. Hence, the problem is to spot the multidegrees R contributing to these vector spaces and to determine the dimensions of $T^p_Y(-R)$. (The minus sign just makes some of the formulas easier.)

(2) Let us return to the trivial example of the A_{n-1}-singularity mentioned in the very beginning. The algebra $A := \mathbb{C}[x,y,z]/(z^n - xy)$ is $\mathbb{Z}^2$-graded via $\deg x := [n,-1]$, $\deg y := [0,1]$, and $\deg z := [1,0]$. The infinitesimal deformations equal $T^1_Y = \mathbb{C}[z]/(z^{n-1})$, and the one-parameter deformation assigned to the element $z^{n-k} \in T^1_Y$ equals $(z^n - xy) + t^{(k)} z^{n-k}$. Substituting $T := z^k + t^{(k)}$, the total space of this deformation is defined by a binomial again, by $Tz^{n-k} - xy$.

The general goal is to look for so-called genuine deformations, i.e. for those which are no longer infinitesimal, but defined over parameter spaces which are reduced or even smooth. To be able to use the language of polyhedral cones, we would like to remain somehow in the category of toric varieties. It turns out to be a good idea to look for deformations having toric total spaces as just seen for A_{n-1}.

(3) The best possible result is the description of the whole versal deformation. This might be done by listing equations, by providing information about its irreducible components, or by different methods.

After a short introduction into the subject of toric varieties we will discuss the progress in each of these questions in a separate section. Emphasis will be put on both cones over projective varieties and toric Gorenstein singularities, i.e. on those toric varieties induced by lattice polytopes.
Our survey does not contain any proofs; the claims are either standard, or easy exercises, or we refer to the original papers.

2 Convex geometry and toric varieties

(2.1) We are going to introduce briefly the notions we need from convex geometry. It should be considered a good opportunity to fix notation, on the one hand, and to get readers from algebraic geometry in the mood for cones and polytopes, on the other hand. References for the details can be found in the new book [Zi] or the appendix in [Od].

Convex cones: Throughout the paper we use the word *cone* for rational, convex, polyhedral cones. If N, M are two mutually dual, free Abelian groups of finite rank, then a cone $\sigma \subseteq N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ can be given either by its fundamental generators

$$\sigma = \langle a^1, \ldots, a^m \rangle := \sum_{i=1}^{m} \mathbb{R}_{\geq 0} \cdot a^i \qquad (a^1, \ldots, a^m \in N)$$

or by finitely many inequalities

$$\sigma = \{a \in N_{\mathbb{R}} \,|\, \langle a, r^j \rangle \geq 0\,;\, j = 1, \ldots, K\} \qquad (r^1, \ldots, r^K \in M)\,.$$

The elements $a^i \in N$ and $r^j \in M$ can be normalized by asking for primitive ones, i.e. which are not proper multiples. (I hope the reader will not be confused by abuse of notation: We use the symbol $\langle \ldots \rangle$ for both the pairing between the mutually dual lattices N, M as well as for indicating the generators of cones; "<" also denotes the face relation.)

The concept of duality interchanges both representations: the cone dual to σ is defined as

$$\sigma^\vee := \{r \in M_{I\!R} \,|\, \langle a, r\rangle \geq 0 \ \text{ for all } a \in \sigma\}\,.$$

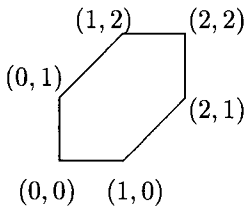
It has $r^1, \ldots, r^K \in M$ as fundamental generators, or it can be given by the inequalities provided by $a^1, \ldots, a^m \in N$.

(2.2) *Polytopes and polyhedra:* Let $(L_{I\!R}, L)$ be a finite-dimensional real vector (or maybe affine) space with a lattice. Rational polyhedra in $(L_{I\!R}, L)$ are given as intersections of finitely many rational half spaces. If additionally compact, they will be called polytopes. A polyhedron is said to be a lattice polyhedron if its vertices are contained in L.

Definition: *For two polyhedra $Q', Q'' \subseteq L_{I\!R}$ we define their Minkowski sum as the polyhedron $Q' + Q'' := \{p' + p'' \,|\, p' \in Q', p'' \in Q''\}$. Obviously, this notion also makes sense for translation classes of polyhedra, hence for affine instead of vector spaces $L_{I\!R}$.*

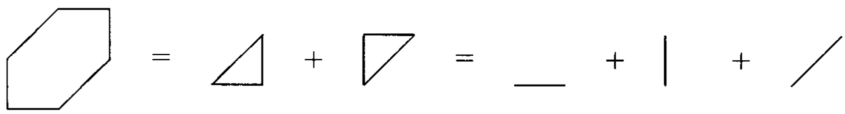
Example: The plane hexagon

$$Q := \operatorname{Conv}\{(0,0), (1,0), (2,1), (2,2), (1,2), (0,1)\} \subseteq I\!R^2$$



hexagon Q

splits into



Every polyhedron $Q \subseteq L_{I\!R}$ is decomposable into the Minkowski sum $Q = Q^c + Q^\infty$ of a (compact) polytope $Q^c \in L_{I\!R}$ and the so-called cone of unbounded directions Q^∞; the latter one is contained in the *vector* space associated to $L_{I\!R}$ which, however, will be identified with $L_{I\!R}$. The cone Q^∞ is uniquely determined by Q, the compact summand is not. However, we can take for Q^c the minimal one – given as the convex hull of the vertices of Q itself. If Q was already compact, then $Q^c = Q$ and $Q^\infty = 0$.

Example: Let $\sigma \subseteq N_{I\!R}$ be a cone, and fix some primitive element $R \in M$. Then $L_{I\!R} := [R = 1] := \{a \in N_{I\!R} \,|\, \langle a, R\rangle = 1\} \subseteq N_{I\!R}$ is an affine space

with lattice $L := [R = 1] \cap N$. We define the crosscut of σ in degree R as the polyhedron $Q := \sigma \cap [R = 1] \subseteq L_{I\!R}$. It has the cone of unbounded directions $Q^\infty = \sigma \cap [R = 0] \subseteq N_{I\!R}$. The compact part Q^c of Q is obtained by describing its vertices: obviously, they correspond exactly to those fundamental generators a^i of σ meeting $\langle a^i, R\rangle \geq 1$ – the actual vertices equal $\bar{a}^i = a^i / \langle a^i, R\rangle$.
Fundamental generators contained in $R^\perp$ can still be "seen" as edges in Q^∞, but those with $\langle \bullet, R\rangle < 0$ are "invisible" in Q. In particular, we can recover the cone σ from Q if and only if $R \in \sigma^\vee$.

(2.3) One of the most frequently used notions will be that of *Minkowski summands* of a given polyhedron $Q \subseteq L_{I\!R}$. Of course, a Minkowski summand Q' of Q should be at least a summand in the usual sense, i.e. there has to be a Q'' such that $Q = Q' + Q''$. However, since $Q = Q' + Q^\infty$ is true for every $Q^c \subseteq Q' \subseteq Q$, this might not be enough; we would like to avoid additional face structure of Q' (not "coming" from Q). We take the following definition from [Sm]:

Definition: *A polyhedron Q' is called a Minkowski summand of Q if there is a Q'' such that $Q = Q' + Q''$ and if $(Q')^\infty = Q^\infty$.*

It is not difficult to see that the faces of $Q' + Q''$ equal the Minkowski sums of the corresponding faces (defined by the same hyperplane in $L_{I\!R}$) of Q' and Q''. In particular, up to dilatation, the set of edges of $Q' + Q''$ equals the union of the corresponding sets for Q' and Q'', respectively. That means, a Minkowski summand has not only the same cone of unbounded directions, but, up to dilatation with a factor ≥ 0, also the same compact edges as the original polyhedron.

This is the moment to describe the "moduli space" of all Minkowski summands of Q, following [Al 3]. After choosing orientations, denote the compact edges of Q by $d^1, \ldots, d^N \in L_{I\!R}$:

Definition: *For every two-dimensional compact face ("two-face") $\varepsilon < Q$ we define its sign vector $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N) \in \{0, \pm 1\}^N$ by*

$$\varepsilon_i := \begin{cases} \pm 1 & \text{if } d^i \text{ is an edge of } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

such that the oriented edges $\varepsilon_i \cdot d^i$ fit to a cycle along the boundary of ε. This determines $\underline{\varepsilon}$ up to sign, and any choice will do. In particular, $\sum_i \varepsilon_i d^i = 0$. Then define $V(Q)$ to be the vector space

$$\{(t_1, \ldots, t_N) \in I\!R^N \mid \sum_i t_i\, \varepsilon_i\, d^i = 0 \ \text{ for every compact two-face } \varepsilon < Q\}.$$

It is obvious that the points of the cone $C(Q) := V(Q) \cap I\!R^N_{\geq 0}$ parametrize the set of Minkowski summands of positive multiples of Q via measuring the dilatation factors of the compact edges. In particular, $\underline{1} \in C(Q)$ corresponds to Q itself.

(2.4) *Affine toric varieties:* Let N, M be two mutually dual, free Abelian groups of finite rank. From now on, the lattice structure of cones and polyhedra becomes important. In particular, isomorphisms between those objects are always assumed to be induced from isomorphisms of the lattices. We are going to describe how to get affine algebraic varieties from convex cones:

Definition: *If $\sigma \subseteq N_{\mathbb{R}}$ is a cone with apex, then we define by $Y_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ the associated, affine toric variety. ($\mathbb{C}[\sigma^\vee \cap M]$ denotes the semigroup ring - obtained by regarding elements $r \in \sigma^\vee \cap M$ as exponents of some "abstract symbol" x.)*
Let $\sigma_1 \subseteq N^1_{\mathbb{R}}$, $\sigma_2 \subseteq N^2_{\mathbb{R}}$ be two cones. Then, a $\mathbb{Z}$-linear map $f : N^1 \to N^2$ such that $f(\sigma_1) \subseteq \sigma_2$ induces an algebraic morphism $f : Y_1 \to Y_2$ in an obvious way. Those maps will be regarded as the morphisms in the category of affine, toric varieties.

The semigroup $S := \sigma^\vee \cap M$ is generated by the finite set E of its irreducible elements. E is often called the Hilbert basis of that semigroup. Assigning to each element $r \in E$ a variable z_r, our affine toric variety Y_σ can be embedded into $\mathbb{C}^E$. It is defined by the binomial equations obtained from "raising" linear dependencies between the r's into the exponents of the z_r's. Just to give an example, the relation $r + 2s = 3t + u$ turns into $z_r\, z_s^2 = z_t^3\, z_u$.

Examples:

(1) The cone $\sigma := \mathbb{R}^k_{\geq 0} \subseteq \mathbb{R}^k$ with $N := \mathbb{Z}^k$ yields $\sigma^\vee \cap M = \mathbb{N}^k$, hence $Y_\sigma = \mathbb{C}^k$.

(2) Let $E \subseteq \sigma^\vee \cap M$ be the Hilbert basis for an arbitrary cone $\sigma \subseteq N_{\mathbb{R}}$. Then, assigning to each $a \in N$ the E-tuple $(\langle a, r\rangle)_{r\in E} \in \mathbb{Z}^E$, defines a $\mathbb{Z}$-linear map $N \to \mathbb{Z}^E$ sending σ into $\mathbb{R}^E_{\geq 0}$. At the level of toric varieties, this yields exactly the embedding $Y_\sigma \hookrightarrow \mathbb{C}^E$ described above.

(3) Let $n, q \in \mathbb{Z}$ be relatively prime numbers. With $(\alpha, \beta) \mapsto (-q\,\alpha + \beta,\, n\,\alpha)$, we obtain a $\mathbb{Z}$-linear map $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ sending $\mathbb{R}^2_{\geq 0}$ onto $\sigma := \langle (1,0), (-q,n)\rangle \subseteq \mathbb{R}^2$. (By the way, any two-dimensional cone can be written in this way.) At the toric level this means we have a morphism $\pi : \mathbb{C}^2 \to Y_\sigma$.
The dual cone equals $\sigma^\vee = \langle [0,1],\, [n,q]\rangle$, and we obtain for $f^*(\sigma^\vee \cap \mathbb{Z}^2)$ the semigroup

$$f^*(\sigma^\vee \cap \mathbb{Z}^2) = \mathbb{N}^2 \cap \{(r_1, r_2) \in \mathbb{Z}^2 \mid r_1 + q\, r_2 \equiv 0 \bmod n\}\,.$$

Hence, the affine coordinate ring $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2]$ of Y_σ equals the subring of $\mathbb{C}[z_1, z_2]$ consisting of polynomials invariant under $\begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$ with ξ

being a primitive n^{th} root of unity. In particular, Y_σ is a cyclic quotient singularity, and π is the quotient map.
For $q = -1$ we obtain the A_{n-1}–singularity mentioned in the very beginning; $n = 4$, $q = 1$ yields Pinkham's example for singularities with reducible versal base space.

We have seen that almost all two-dimensional cones yield singular toric varieties. This reflects the general situation - smooth, affine toric varieties are boring: If σ is a top-dimensional cone, then Y_σ *is smooth if and only if* σ *is a simplex generated by a* $\mathbb{Z}$*-basis of* N, i.e. the determinant of its fundamental generators has to be ± 1. Then, Y_σ is isomorphic to the affine space.
Toric varieties got their name because they always contain the torus $T = \operatorname{Spec} \mathbb{C}[M]$. This algebraic group acts on them and causes a (finite) stratification into T-orbits. The unique closed orbit in an affine toric variety (if σ is top-dimensional, then it is a point) is the most singular one. In higher dimensions, most cones are no longer simplicial. This means that the singularities get worse than quotient singularities.

(2.5) *General toric varieties:* As already mentioned, morphisms between affine toric varieties arise from $\mathbb{Z}$-linear maps $f : N^1 \to N^2$ such that $f(\sigma_1) \subseteq \sigma_2$. A very important special case is where $f : N \to N$ is the identity map and σ_1 is a face of σ_2. If $r \in \sigma_2^\vee \cap M$ actually cuts out this face (i.e. $\sigma_1 = \sigma_2 \cap r^\perp$), then $\mathbb{C}[\sigma_2^\vee \cap M]$ equals the localization of $\mathbb{C}[\sigma_1^\vee \cap M]$ by the element x^r. In particular, the induced map $Y_{\sigma_1} \to Y_{\sigma_2}$ is an open embedding identifying the first variety with the open subset $[x^r \neq 0] \subseteq Y_{\sigma_1}$. Moreover, every open embedding in our category arises that way.

Definition: *If* Σ *is a fan in* $N_{\mathbb{R}}$ *(i.e. a finite collection of cones such that* $\sigma, \tau \in \Sigma$ *always implies* $\tau \cap \sigma \leq \tau, \sigma$*, and such that* Σ *contains with every cone all of its faces), then the toric variety* Y_Σ *is obtained by gluing together the affine pieces* Y_σ $(\sigma \in \Sigma)$ *along common faces of* Σ*-cones.*
A map between toric varieties $Y_{\Sigma^1}, Y_{\Sigma^2}$ *is given by a* $\mathbb{Z}$*-linear* $f : N^1 \to N^2$ *such that for each* $\sigma_1 \in \Sigma^1$ *there is some* $\sigma_2 \in \Sigma^2$ *meeting* $f(\sigma_1) \subseteq \sigma_2$.

It is well known that complete toric varieties arise from fans with $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma = N_{\mathbb{R}}$. If, moreover, Σ is the inner normal fan of some lattice polytope P in M, then $Y_P := Y_\Sigma$ is projective. The polytope itself reflects the projective embedding; the lattice points of P correspond in an easy way to a natural basis of the global sections of an ample line bundle $\mathcal{L}_P$.
More generally, there is a one-to-one correspondence between certain lattice polytopes, on the one hand, and globally generated invertible sheaves on Y_Σ, on the other; Minkowski addition translates into the tensor product. Hence, it seems quite natural that, for projective toric varieties, the vector space $V(P)$ of (2.3) appears as $V(P) = \operatorname{Pic} Y_P \otimes_{\mathbb{Z}} \mathbb{R}$.

Examples:

(1) The k-dimensional fan spanned by the canonical basis vectors $e^1, \dots, e^k$ of $\mathbb{Z}^k$ and $-e$, with $e := e^1 + \cdots + e^k$, defines the projective space $\mathbb{P}^k$.

(2) The subdivision of the smooth cone $\sigma = \mathbb{R}^k_{\geq 0} = \langle e^1, \dots, e^k \rangle$ into the union of $\sigma_i := \langle e^1, \dots, \hat{e}^i, \dots, e^k, e \rangle$ $(i = 1, \dots, k)$ describes the blowing up of the origin in the affine k-space.

(3) If $\sigma = \langle a^1, \dots, a^m \rangle$ is an arbitrary cone (with apex), then the normalized blowing up of Y_σ in the closed orbit is given by the subdivision of σ into the union of $\sigma_r := \{a \in \sigma \,|\, \langle a, r \rangle \leq \langle a, E \rangle\}$ with r running through the Hilbert basis E of $\sigma^\vee \cap M$.

In general, every fan can be subdivided into a "smooth" fan. That means, every toric variety admits a toric desingularization.

(2.6) Let $P \subseteq (L_{\mathbb{R}}, L)$ be a lattice polytope. Embedding P via $P \subseteq L_{\mathbb{R}} \cong L_{\mathbb{R}} \times \{1\} \hookrightarrow L_{\mathbb{R}} \times \mathbb{R} =: M_{\mathbb{R}}$ into the next dimension, we obtain rational, polyhedral cones $\sigma^\vee := \operatorname{cone}(P) := \mathbb{R}_{\geq 0} \cdot P \subseteq M_{\mathbb{R}}$ and $\sigma := \sigma^{\vee\vee} \subseteq N_{\mathbb{R}}$. Since

$$\mathbb{C}[S] := \mathbb{C}[\sigma^\vee \cap M] = \oplus_{d \geq 0} \mathbb{C}[d\,P \cap M] = \oplus_{d \geq 0} H^0(Y_P, \mathcal{L}_P^{\otimes d}),$$

the toric variety Y_σ equals the *affine cone over the projective variety* $(Y_P, \mathcal{L}_P)$. There is a distinguished point $a^* := (\underline{0}, 1) \in \sigma \subseteq N_{\mathbb{R}}$; the equation $[a^* = 1]$ recovers P from $\sigma^\vee$. Moreover, a^* may be used to make the ring $\mathbb{C}[\sigma^\vee \cap M]$ $\mathbb{Z}$-graded $(\deg x^r := \langle a^*, r \rangle)$, and we obtain $Y_P = \operatorname{Proj} \mathbb{C}[\sigma^\vee \cap M]$ while $Y_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$.
We may elucidate the relation between Y_P and its affine cone Y_σ also from another point of view: the open subset $Y_\sigma \setminus \{0\} \subseteq Y_\sigma$ is given by the fan $\partial\sigma$ contained in $N_{\mathbb{R}}$, and the projection $\pi : Y_\sigma \setminus \{0\} \to Y_P$ is provided by the $\mathbb{Z}$-linear map $\pi : N \twoheadrightarrow N/\mathbb{Z} \cdot a^* = L_{\mathbb{R}}{}^*$. It sends proper faces of σ isomorphically onto Σ-cones, meaning that $Y_\sigma \setminus \{0\}$ is a $\mathbb{C}^*$-bundle over Y_P.

(2.7) Finally, we would like to explain Ishida's relation (cf. [Ish], Theorem 7.7.) between lattice polytopes and Gorenstein singularities: up to sign, the differential form $\omega_0 := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_k}{x_k}$ on the torus $(\mathbb{C}^*)^k \cong \operatorname{Spec}\mathbb{C}[M] = Y_0 \subseteq Y_\sigma$ does *not* depend on the special choice of coordinates. Multiples $x^r \cdot \omega_0$ are holomorphic on Y_σ if and only if r belongs to the interior of $\sigma^\vee$. Since Y_σ is normal, this means that we can describe the canonical module as $\omega_Y = \big(\oplus_{r \in (\operatorname{int} \sigma^\vee) \cap M} \mathbb{C} \cdot x^r \big) \cdot \omega_0$. In particular, Y_σ is Gorenstein, i.e. ω_Y is invertible if and only if there is an $R^* \in M$ such that $(\operatorname{int} \sigma^\vee) \cap M = R^* + (\sigma^\vee \cap M)$. Replacing ω_Y by its g-th tensor power, we have obtained the following criterion:

Let $Y_\sigma = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$ be an affine toric variety given by a cone $\sigma = \langle a^1, \ldots, a^m\rangle$. Then, Y is $\mathbb{Q}$-Gorenstein if and only if there is a primitive element $R^* \in M$ and a natural number $g \in \mathbb{N}$ such that

$$\langle a^i,\, R^*\rangle = g \quad \textit{for each } i = 1, \ldots, m.$$

Y is Gorenstein if and only if, additionally, $g = 1$.

In particular, toric Gorenstein singularities are obtained by putting a lattice polytope $Q \subseteq (L_{\mathbb{R}}, L)$ into the affine hyperplane $L_{\mathbb{R}} \times \{1\} \subseteq L_{\mathbb{R}} \times \mathbb{R} =: N_{\mathbb{R}}$ and defining $\sigma := \operatorname{cone}(Q) = \mathbb{R}_{\geq 0} \cdot Q$. The canonical degree R^* equals $[\underline{0}, 1]$ in this setting. As an example, lattice intervals of length n provide the two-dimensional A_{n-1}-singularities; see (2.4)(3).

3 Infinitesimal deformations

(3.1) If $Y \subseteq \mathbb{C}^w$ is defined by an ideal $I = (f_1, \ldots, f_s) \subseteq \mathbb{C}[z_1, \ldots, z_w]$, then we denote by $A := \mathbb{C}[\underline{z}]/I$ the algebra of regular functions, by $\mathcal{R} \subseteq \mathbb{C}[\underline{z}]^s$ the $\mathbb{C}[\underline{z}]$-module of linear relations between $f_1, \ldots, f_s$, and by $\mathcal{R}_0 \subseteq \mathcal{R}$ the so-called Koszul relations generated by all $f_j e^k - f_k e^j \in \mathbb{C}[\underline{z}]^s$. In particular, we have the exact sequences

$$0 \to \mathcal{R} \to \mathbb{C}[\underline{z}]^s \to I \to 0 \quad \text{and} \quad 0 \to I \to \mathbb{C}[\underline{z}] \to A \to 0\,.$$

Then, the easiest way to define the vector spaces T^1_Y and T^2_Y is

$$T^1_Y := {\operatorname{Hom}_A(I/I^2, A)} \Big/ {\operatorname{Hom}_A(A^w, A)}$$

via

$$d:\ I/I^2 \to A^w \quad \text{with} \quad d(f_j) := \Big(\frac{\partial f_j}{\partial z_1}, \ldots \frac{\partial f_j}{\partial z_w}\Big)$$

and

$$T^2_Y := {\operatorname{Hom}_{\mathbb{C}[z]}(\mathcal{R}/\mathcal{R}_0, A)} \Big/ {\operatorname{Hom}_{\mathbb{C}[z]}(\mathbb{C}[\underline{z}]^s, A)}\ .$$

These definitions seem to depend on the embedding of Y, but they do not. In the above straightforward formulas, the relation of the vector spaces T^p_Y to deformation theory becomes apparent. For instance, if $\xi \in \operatorname{Hom}_A(I/I^2, A)$, then the infinitesimal deformation represented by ξ may be obtained by replacing f_j with the perturbed equation $f_j + \varepsilon\, \xi(f_j)$.

(3.2) A more fancy way to obtain T^1_Y and T^2_Y is to consider them as the first and second André-Quillen cohomology groups, respectively. This cohomology

theory is obtained from the cotangent complex which is defined for any $\mathbb{C}$-algebras; it is closely related to Hochschild and Harrison cohomology. A nice introduction into this subject may be found in [Lo], §3.5 - §4.5.
To calculate the vector spaces T^p_Y for affine toric varieties Y_σ, the Harrison cohomology approach was most successful. It yields the following results (cf. [AlSl]):

The T^p_Y admit an M-(multi) grading as does any other natural module over $A = \mathbb{C}[\sigma^\vee \cap M]$; let us fix an element $R \in M$. For any face $\tau \le \sigma$ we define

$$K^R_\tau := \left[\sigma^\vee \cap (R - \operatorname{int} \tau^\vee) \cap M\right] \setminus \{0\}\,.$$

Definition: Let $K \subseteq M$ be an arbitrary subset of the lattice M. A function $f : K \to \mathbb{C}$ is called *quasilinear* if $f(r) + f(s) = f(r+s)$ for any r and s with $r, s, r+s \in K$. The vector space of quasilinear functions is denoted by $\overline{\operatorname{Hom}}\,(K, \mathbb{C})$.

The sets K^R_τ admit the following properties:

(i) $K^R_0 = (\sigma^\vee \cap M) \setminus \{0\}$, and $K^R_i := K^R_{a^i} = \{r \in K_0 |\; \langle a^i, r\rangle < \langle a^i, R\rangle\}$ with $i = 1, \ldots, m$ are "thick strips" along the facets of $\sigma^\vee$.

(ii) For $\tau \ne 0$ the equality $K^R_\tau = \bigcap_{a^i \in \tau} K^R_i$ holds. Moreover, if σ is a top-dimensional cone, $K^R_\sigma = K_0 \cap (R - \operatorname{int} \sigma^\vee)$ is a (diamond shaped) finite set.

(iii) The dependence of the sets K^R_τ on τ is a contravariant functor. This gives rise to the complex $\overline{\operatorname{Hom}}\,(K^R_\bullet, \mathbb{C})$ with

$$\overline{\operatorname{Hom}}\,(K^R_p, \mathbb{C}) := \oplus_{\tau \le \sigma, \dim \tau = p} \overline{\operatorname{Hom}}\,(K^R_\tau, \mathbb{C}) \qquad (0 \le p \le \dim \sigma)$$

and the usual differentials.

(iv) If $\tau \le \sigma$ is a smooth face, then the injections $\operatorname{Hom}(\operatorname{span}_{\mathbb{C}} K^R_\tau, \mathbb{C}) \hookrightarrow \overline{\operatorname{Hom}}\,(K^R_\tau, \mathbb{C})$ are also isomorphisms. Moreover,

$$\operatorname{span}_{\mathbb{C}} K^R_\tau = \bigcap_{a^i \in \tau} \operatorname{span}_{\mathbb{C}} K^R_i,$$

and the latter vector spaces equal $\operatorname{span}_{\mathbb{C}} K^R_i = M_{\mathbb{C}}$, $(a^i)^\perp$, or 0 if $\langle a^i, R\rangle \ge 2$, $= 1$, or ≤ 0, respectively.

Now we can express the André-Quillen cohomology groups T^p_Y in terms of the sets K^R_τ:

Theorem: ([AlSl])
1) Let σ be an arbitrary rational, polyhedral cone with apex in 0. Then, for every $R \in M$,