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GEOMETRY OF SETS AND MEASURES IN EUCLIDEAN SPACES

Fractals and rectifiability

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To Vappu

Contents

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Basic notation

We introduce here the notation for some basic concepts which are not defined in the text. A more extensive glossary of notation is given at the end of the book.

$\mathbf{Z}$, the set of integers.

$\mathbf{R}$, the set of real numbers.

$\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$.

$\mathbf{C}$, the set of complex numbers.

$\overline{z}$, $\operatorname{Re} z$ and $\operatorname{Im} z$ are the complex conjugate, real part and imaginary part of $z \in \mathbf{C}$.

$\mathbf{R}^n$, the n-dimensional euclidean space equipped with the inner product $x \cdot y$ and norm $|x|$.

$S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$, the unit sphere.

$[a, b]$, (a, b), $[a, b)$ and $(a, b]$ are the closed, open and half-open intervals in $\overline{\mathbf{R}}$ with end-points $a, b \in \overline{\mathbf{R}}$.

$\mathcal{L}^n$, the Lebesgue measure on $\mathbf{R}^n$.

$\alpha(n) = \mathcal{L}^n\{x \in \mathbf{R}^n : |x| \leq 1\}$, the volume of the unit ball.

$\overline{A} = \operatorname{Cl} A$, the closure of the set A.

∂A, the boundary of A.

χ_A, the characteristic function of A.

$A + B = \{x + y : x \in A,\ y \in B\}$.

$A + a = \{x + a : x \in A\}$.

$\operatorname{card} A$, the number points in the set A; possibly 0 or ∞.

$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$, the union of the set family $\mathcal{A}$.

$\bigcap \mathcal{A}$, the intersection of $\mathcal{A}$.

We often use notation like $\{x : f'(x) > 0\}$ to mean the set of those points x where the derivative $f'(x)$ *exists* and is positive.

The symbol $\square$ denotes the end of the proof.