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**GEOMETRY OF SETS AND MEASURES IN
EUCLIDEAN SPACES**

Fractals and rectifiability

Pertti Mattila

University of Jyväskylä, Finland



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Rectifiability - Pertti Mattila
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PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge CB2 1RP

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK <http://www.cup.cam.ac.uk>
40 West 20th Street, New York, NY 10011-4211, USA <http://www.cup.org>
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

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Parts of this work were first published by Universidad Extremadura in 1986
This edition first published 1995
First paperback edition 1999

A catalogue record for this book is available from the British Library

ISBN 0 521 46576 1 hardback
ISBN 0 521 65595 1 paperback

Transferred to digital printing 2004

Cambridge University Press

0521655951 - Geometry of Sets and Measures in Euclidean Spaces: Fractals and
Rectifiability - Pertti Mattila

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Acknowledgements

This book grew out of the lecture notes Mattila [12] which were based on the lectures on geometric measure theory that I gave in Jarandilla de la Vera in 1984 at a summer school organized by Asociación Matemática Española and Universidad de Extremadura. I renew my thanks to Miguel de Guzmán and the other organizers of this meeting as well as to the inspiring audience. The preparation of this book was also greatly influenced by the course I gave as a visitor of Centre de Recerca Matemàtica at Universitat Autònoma de Barcelona in the spring of 1992. I want to thank the Centre for its hospitality and financial support; in particular my thanks are due to Joaquim Bruna, Manuel Castellet and Joan Verdera, and again to the active participants of the lectures. I am much obliged to Kenneth Falconer, Maarit Järvenpää and David Preiss, who corrected many mistakes and suggested numerous improvements in the first versions of the manuscript. Several other mathematicians have made useful comments that have been of great help to me. In particular I am grateful for this to Guy David, Tero Kilpeläinen, Peter Möllers, Joan Orobítg, Yuval Peres and Stephen Semmes. For skilful typing with \TeX I want to thank Eira Henriksson and Marja-Leena Rantalainen, and for other assistance Ari Lehtonen. Finally I would like to thank David Tranah and others from the Cambridge University Press for their help in the production of the book.

For financial support I am indebted to the Academy of Finland in different forms and during long periods. Parts of this book were written during the fall term 1991 at Stanford University and at the Institute for Advanced Study in Princeton, and during May–June 1992 at Institut des Hautes Etudes Scientifiques in Bures-sur-Yvette; I acknowledge with gratitude the financial support and the fruitful atmosphere of these institutes.

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Basic notation

We introduce here the notation for some basic concepts which are not defined in the text. A more extensive glossary of notation is given at the end of the book.

\mathbf{Z} , the set of integers.

\mathbf{R} , the set of real numbers.

$\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$.

\mathbf{C} , the set of complex numbers.

\bar{z} , $\operatorname{Re} z$ and $\operatorname{Im} z$ are the complex conjugate, real part and imaginary part of $z \in \mathbf{C}$.

\mathbf{R}^n , the n -dimensional euclidean space equipped with the inner product $x \cdot y$ and norm $|x|$.

$S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$, the unit sphere.

$[a, b]$, (a, b) , $[a, b)$ and $(a, b]$ are the closed, open and half-open intervals in $\overline{\mathbf{R}}$ with end-points $a, b \in \overline{\mathbf{R}}$.

\mathcal{L}^n , the Lebesgue measure on \mathbf{R}^n .

$\alpha(n) = \mathcal{L}^n\{x \in \mathbf{R}^n : |x| \leq 1\}$, the volume of the unit ball.

$\overline{A} = \operatorname{Cl} A$, the closure of the set A .

∂A , the boundary of A .

χ_A , the characteristic function of A .

$A + B = \{x + y : x \in A, y \in B\}$.

$A + a = \{x + a : x \in A\}$.

$\operatorname{card} A$, the number points in the set A ; possibly 0 or ∞ .

$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$, the union of the set family \mathcal{A} .

$\bigcap \mathcal{A}$, the intersection of \mathcal{A} .

We often use notation like $\{x : f'(x) > 0\}$ to mean the set of those points x where the derivative $f'(x)$ *exists* and is positive.

The symbol \square denotes the end of the proof.