Cambridge University Press
978-0-521-65588-0 - Groups St Andrews 1997 in Bath, I
Edited by C. M. Campbell, E. F. Robertson, N. Ruskuc and G. C. Smith
Excerpt
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RADICAL RINGS AND PRODUCTS OF GROUPS

BERNHARD AMBERG* and YAROSLAV SYSAK†[1]

*Fachbereich Mathematik der Universität Mainz, D-55099 Mainz, Germany
†Institute of Mathematics, Ukrainian National Academy of Sciences, 252601 Kiev, Ukraine

1 Introduction

Groups which can be written as a product $G = AB$ of two of its subgroups A and B have been studied by many authors; see for instance the monograph [8]. In these investigations groups of the form $G = AB = AM = BM$ where M is a normal subgroup of G play a particular role. If M is abelian and the intersections $A \cap M$ and $B \cap M$ are trivial, there is an interesting connection with radical rings (in sense of Jacobson) and their generalisations to so-called radical modules. In the following we will discuss in detail some aspects of this connection and, in addition, consider certain structural questions about radical rings.

The notation is standard and can be found in [36] and [37] for the group-theoretical terminology and in [27] and [40] for the ring-theoretical terms.

2 Fundamentals

2.1 Triply factorized groups

Consider a group $G = AB$ which is the product of two subgroups A and B. To study such groups it is desirable to find subgroups of G which are likewise the product of a subgroup of A with a subgroup of B. We recall some elementary facts, which can be found in [8], Section 1.1.

A subgroup S of the group $G = AB$ is called *factorized* if $S = (A \cap S)(B \cap S)$ and $A \cap B \subseteq S$. It is easy to see that a subgroup S of $G = AB$ is factorized if and only if, whenever $ab \in S$ with $a \in A$ and $b \in B$, then $a \in S$. In particular, the group $G = AB$ itself is factorized.

Obviously the intersection of an arbitrary set of factorized subgroups of $G = AB$ is a factorized subgroup of G. Therefore, if U is a subgroup of $G = AB$, then there exists a smallest factorized subgroup $X(U)$ of G containing U; this subgroup $X(U)$ is called the *factorizer* of U in G. Clearly, a subgroup U of $G = AB$ is factorized if and only if $U = X(U)$.

If N is a normal subgroup of $G = AB$, then $X(N) = AN \cap BN = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN)$ (see [8], Lemma 1.1.4). This means that in many investigations of factorized groups one has to consider *triply factorized* groups of the form $G = AB = AM = BM$ where M is a normal subgroup of G. If M is

[1]The second author likes to thank the Deutsche Forschungsgemeinschaft for financial support and the Department of Mathematics of the University of Mainz for its excellent hospitality during the preparation of this paper.

abelian, then $C = (A \cap M)(B \cap M)$ is normal in G. Factoring out this normal subgroup we arrive at a triply factorized group

$$G = AB = AM = BM \text{ with the "intersection property" } A \cap M = B \cap M = 1.$$

In this case A and B are complements of M in the factorized group $G = AB$, and M is a $\mathbb{Z}A$-module with special properties. Note that A and B are not conjugate unless $A = B = G$.

2.2 Radical rings

Let R be an associative ring, not necessarily with an identity element. The set of all elements of R forms a semigroup with identity element $0 \in R$ under the operation $a \circ b = a + b + ab$ for all a and b of R. The group of all invertible elements of this semigroup is called the *adjoint group* of R and denote by R°. Following Jacobson ([27], p. 4) a ring R is called *radical* if $R = R^\circ$, which means that R coincides with its Jacobson radical. Obviously such a ring does not have an identity element. If the radical ring R is embedded in the usual way into a ring R_1 with identity 1, then R° is isomorphic with the subgroup $1 + R$ of the group of units of R_1. It is easy to see that this property even characterizes radical rings. It should be also noted that subrings of a radical ring need not be radical, but right and left ideals, homomorphic images and cartesian products of radical rings are always radical. The next theorem gives an other important property of radical rings (see [27], p. 11).

Theorem 2.1 *If R is a radical ring, then the ring $M_n(R)$ of all $n \times n$ matrices over R is likewise radical.*

A ring R is a *nil ring* if every element a of R is *nilpotent*, i.e. there exists a positive integer $n = n(a)$ such that $a^n = 0$. It is clear that every nil ring is radical, and it is not difficult to see that a radical ring is nil if and only if each of its subrings is radical.

Example 2.2 For every prime p let W_p be the set of all rational numbers whose numerator is divisible by p, but not its denominator. Then W_p is a radical ring, as $1 + W_p$ is a group. But W_p has no non-trivial nilpotent elements, and so is not nil. Moreover, every finitely generated subring of W_p is not radical.

For each positive integer n let R^n be a subring of R which is generated by all products of n elements of R. Clearly R^n is an ideal of R. A ring R is *nilpotent* if $R^{n+1} = 0$ for some non-negative integer n; the smallest such n is called the *class* of R. For instance, for every prime p and every natural number n the ring $p\mathbb{Z}/p^{n+1}\mathbb{Z}$ is nilpotent of class n. Trivial examples of nilpotent rings are the *null rings*, i.e. those of class 1. A proper subclass of the class of nil rings which contains all nilpotent rings is the class of *locally nilpotent* rings, in which every finitely generated subring is nilpotent. The first example of a nil ring which is not locally nilpotent was given by Golod (see for instance [40], Theorem 6.2.9). Properly contained between the nilpotent rings and the locally nilpotent rings is the class of *T-nilpotent* rings, in which every non-trivial homomorphic image has a non-trivial annihilator. The direct sum of the nilpotent rings $p\mathbb{Z}/p^{n+1}\mathbb{Z}$ for every non-negative integer n is a T-nilpotent ring which is not nilpotent. Observe that a ring R is nilpotent, T-nilpotent or locally nilpotent if and only if for each positive integer n the matrix ring $M_n(R)$ has the same property, respectively. However it is unknown at present *whether for a nil ring R the ring $M_2(R)$ is also nil.* This question is equivalent to the famous problem of Koethe that every nil one-sided ideal of a ring is contained in a nil ideal of this ring (see [41]).

The relation among all above-mentioned classes of rings can be seen from the following list, where each class is a proper subclass of the preceding one.

- radical rings
- nil rings
- locally nilpotent rings
- T-nilpotent rings
- nilpotent rings
- null rings

Observe also that if a ring R is nilpotent of class n, T-nilpotent or locally nilpotent, then its adjoint group R° is nilpotent of class at most n, hypercentral or locally nilpotent, respectively, and if R is a null ring, then R° is isomorphic with its additive group R^+. As Example 2.2 shows, the converse of this statement is false.

2.3 Construction of triply factorized groups

Radical rings may be used to construct examples of triply factorized groups in the following way (see [45] and [8], Section 6.1).

Let P be a right ideal of the radical ring R and let $M = R/P$ as a right R-module. The adjoint group $A = R^\circ$ operates on M via the rule $m^a = m + ma$ for a in A and m in M. The *associated group* $G(M) = A \ltimes M$ is the semidirect product of M by A. If $B = \{am \mid m = a + P,\ a \in A\}$ is the diagonal of $G(M)$, then B is a subgroup of $G(M)$ and

$$G = G(M) = A \ltimes M = B \ltimes M = AB.$$

Moreover, B is isomorphic to R° and the intersection $A \cap B$ is isomorphic to P°. If in particular $P = 0$, then the normal subgroup M of $G(M)$ is isomorphic to the additive group R^+ of R and $A \cap B = 1$. In this case, the group $G(R)$ can also be represented as a matrix group over R.

Indeed, consider the ring of 2×2-matrices of the form $\begin{pmatrix} 0 & R \\ 0 & R \end{pmatrix}$ over a radical ring R. This is a left ideal of the matrix ring $M_2(R)$ and so is a radical ring by Theorem 2.1. If this ring is regarded as a subring of the matrix ring $M_2(R_1)$, then the adjoint group of $M_2(R)$ is isomorphic with the multiplicative subgroup

$$\Gamma(R) = \begin{pmatrix} 1 & R \\ 0 & 1+R \end{pmatrix}$$

of the group of units of $M_2(R_1)$. It is easy to see that the group $\Gamma(R)$ is isomorphic to $G(R)$, its subgroups $A_R = \begin{pmatrix} 1 & 0 \\ 0 & 1+R \end{pmatrix}$ and $M_R = \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}$ are isomorphic to A and M, respectively, and the subgroup B_R of $\Gamma(R)$ consisting of all 2×2-matrices of the form $\begin{pmatrix} 1 & r \\ 0 & 1+r \end{pmatrix}$ where $r \in R$ is isomorphic to B.

We mention some elementary relations between certain properties of the ring R and those of the group $\Gamma(R)$. Observe first that if S is a radical subring of R, then $\Gamma(S)$ is a subgroup of $\Gamma(R)$. Moreover, a subgroup G of $\Gamma(R)$ has the decomposition $G = AB = AM = BM$ with $A = A_R \cap G$, $B = B_R \cap G$ and $M = M_R \cap G$, if and only if $G = \Gamma(S)$ for some radical subring S of R.

Lemma 2.3 *Let S be a radical subring of a radical ring R. Put $A_S = A_R \cap \Gamma(S)$, $B_S = B_R \cap \Gamma(S)$ and $M_S = M_R \cap \Gamma(S)$. Then the following statements are valid.*

1) $\Gamma(S) = A_S B_S = A_S M_S = B_S M_S$.
2) *The subring S is a right, left or two-sided ideal of R if and only if M_S is normal in $\Gamma(R)$, $\Gamma(S)$ is normal in $A_S M_R$ or $\Gamma(S)$ is normal in $\Gamma(R)$, respectively.*
3) *If ${}^{\perp}R$ and $R^{\perp}$ are the left and right annihilators of R respectively, then $C_{M_R}(A_R) = \begin{pmatrix} 1 & {}^{\perp}R \\ 0 & 1 \end{pmatrix}$ and $C_{A_R}(M_R) = \begin{pmatrix} 1 & 0 \\ 0 & 1+R^{\perp} \end{pmatrix}$. In particular, if R has no divisors of zero, then $C_{A_R}(M_R) = C_{M_R}(A_R) = 1$.*
4) *The ring R is nilpotent of class n or locally nilpotent if and only if the group $\Gamma(R)$ is nilpotent of class n or locally nilpotent, respectively.*
5) *The ring R is a Fitting ring, i.e. R is a sum of its nilpotent ideal, if and only if the group $\Gamma(R)$ is a product of nilpotent normal subgroups of the form $G = AB = AM = BM$ where $A = A_R \cap G$, $B = B_R \cap G$ and $M = M_R \cap G$.*

Conversely, if $G = AB = AM = BM$ is a triply factorized group with three abelian subgroups A, B and M where M is normal in G and $A \cap B = A \cap M = B \cap M = 1$, then there exist a commutative radical ring R and an isomorphism α from G onto $\Gamma(R)$ such that $\alpha(A) = A_R$, $\alpha(M) = M_R$ and $\alpha(B) = B_R$ (for details see [8], Proposition 6.1.4). This result cannot immediately be extended to triply factorized groups with nilpotent subgroups A, B and M, as the following example shows.

Example 2.4 Let M be an elementary abelian group of order 8 generated by elements m_1, m_2, m_3 of order 2, and let A be a dihedral group of order 8 generated by elements a and b such that $a^4 = b^2 = (ab)^2 = 1$. The operation of A on M be as follows:

$$m_1{}^a = m_1{}^b = m_1, m_2{}^a = m_1 m_2, m_2{}^b = m_2, m_3{}^a m_2 m_3, m_3{}^b = m_1 m_2 m_3.$$

Let B be the subgroup of the semidirect product $G = AM$ of M by A generated by the elements am_3 and bm_2. Then $G = AB = AM = BM$ is a nilpotent group with $A \cap B = A \cap M = B \cap M = 1$ and there exists no radical ring R such that G is isomorphic to $\Gamma(R)$.

Indeed, assume that there exists such a radical ring R. It is well-known that the Jacobson radical of a ring with minimal condition on left or right ideals is nilpotent (see [27], p. 38). This implies in particular that the finite radical ring R is nilpotent and so $R^{\perp} \neq 0$. Therefore $C_A(M) \neq 1$ by Lemma 2.3.3. On the other hand, direct computation shows that $C_A(M) = 1$, a contradiction.

2.4 Examples

Some examples of triply factorized groups with additional properties can be constructed for various radical rings and can be found in [45] and [8]. Their properties follow from those of the radical rings under consideration and from Lemma 2.3.

First, using the Jacobson radical of the algebra $F[[x]]$ of formal power series in an indeterminate x over the field F of p elements, which coincides with the ideal $xF[[x]]$ (see [27], p. 21), as an example of a radical F-algebra, we obtain the following.

Example 2.5 There exists a metabelian group G with the following properties.

(i) $G = AB = AM = BM$ where A and B are torsion-free abelian subgroups and M is an infinite elementary abelian normal p-subgroup for some prime p.

(ii) $A \cap B = A \cap M = B \cap M = 1$.

(iii) 1 is the only normal subgroup of G contained in A or B.

Next, let R be the augmentation ideal of the group algebra FA of a Prüfer p-group A over the field F with p elements. Then R is a nil F-algebra (see for instance [33], Lemma 8.1.17) whose additive group R^+ is an infinite elementary abelian p-group and the adjoint group R° is a divisible abelian p-group of infinite rank. For every r of R there is an s in R such that $1 + r = (1 + s)^p = 1 + s^p$ and so $r = s^p = ss^{p-1} \in R^2$. Thus $R = R^2$. It easy to see that the ideal rFA of R is nilpotent for every r of R and so R is generated by its nilpotent ideals. This leads to the following.

Example 2.6 There exists a metabelian p-group G with the following properties.

(i) $G = AB = AM = BM$ where A and B are divisible abelian subgroups and $M = G'$ is an elementary abelian normal subgroup of G.

(ii) $A \cap B = A \cap M = B \cap M = 1$.

(iii) G has trivial centre and is generated by its nilpotent normal subgroups.

(iv) 1 is the only normal subgroup of G contained in A or B.

(v) There is no proper normal subgroup of G containing A or B.

Finally, for every prime p there exists a commutative nil ring R of characteristic p, whose adjoint and additive groups are both elementary abelian p-groups, and such that $R^2 = R$. This leads to the following example (see [8], Theorem 6.1.3).

Example 2.7 (Holt and Howlett [26]) For every prime p there exists a countable locally finite group G of exponent p^2 with the following properties.

(i) $G = AB = AM = BM$, where A, B, and M are elementary abelian p-subgroups and M is normal in G.

(ii) $A \cap B = A \cap M = B \cap M = 1$.

(iii) G has trivial centre and is generated by its nilpotent normal subgroups.

(iv) 1 is the only normal subgroup of G contained in A or B.

(v) There is no proper normal subgroup of G containing A or B.

2.5 Adjoint groups factorized by three pairwise permutable subgroups

Let R be a radical ring and R_1 a ring obtained from R by adjoining an identity 1. Denote by $\Gamma_2(R)$ the group of all invertible 2×2-matrices over R_1 that are congruent to the identity matrix modulo R, and by $t_{ij}(r)$ the transvection with the element r in the position (i, j). In fact,

$$\Gamma_2(R) = \begin{pmatrix} 1+R & R \\ R & 1+R \end{pmatrix}.$$

It is easy to see that this group is isomorphic to the adjoint group of the matrix ring $M_2(R)$ which is radical by Theorem 2.1. The following assertion can be verified directly.

Proposition 2.8 (Sysak [46]) *Let A be the subgroup of $\Gamma_2(R)$ consisting of all its diagonal matrices, $B = t_{12}(-1)At_{12}(1)$ and $C = t_{21}(-1)At_{21}(1)$. Then $\Gamma_2(R) = ABC$ and the subgroups A, B, C commute pairwise, i.e. $AB = BA$, $AC = CA$ and $BC = CB$.*

It is clear that if the ring R is commutative, then the subgroups A, B and C of $\Gamma_2(R)$ are abelian. Hence, as a consequence of Proposition 2.8 we obtain the existence of insoluble linear groups over a field that are the product of three pairwise permutable abelian subgroups (see [46]). Note that every group which is a product of two abelian subgroups must be metabelian by a well-known theorem of Ito (see [8], Theorem 2.1.1).

Corollary 2.9 *Let F be a field whose multiplicative group is non-periodic. Then in F there exists a radical subring $R \neq 0$, and the group $\Gamma_2(R)$ contains a non-abelian free subgroup. In particular, $\Gamma_2(R)$ is an insoluble linear group over F which is a product of three pairwise permutable abelian subgroups.*

The next example follows from Proposition 2.8 and shows that for every prime p and each natural number d there exists a finite p-group that is decomposable into a product of three pairwise permutable cyclic subgroups and whose derived length is greater than d.

Example 2.10 Let p be a prime and $R = p\mathbb{Z}_{p^m}$ for the odd p or $R = 4\mathbb{Z}_{2^m}$ for $p = 2$ where $m \geq 3$. Then the factor group $\Gamma_2(R)/Z(\Gamma_2(R))$ is a product of three pairwise permutable cyclic subgroups of orders p^{m-1} or 2^{m-2}, respectively, and its derived length is at least $log_2(m)$.

2.6 Some general questions

The above construction raises the following questions:

(1) What can be said about the structure of the adjoint group of a radical ring?

(2) Which relations exist among the groups R°, R^+, and $G(R)$ for a radical ring R?

(3) How can these results be used in the study of factorized groups?

(4) Which triply factorized groups with special properties can be constructed using various radical rings?

(5) Which results about factorized groups can be used to study radical rings?

3 The structure of radical rings

3.1 Finiteness conditions

The following finiteness conditions play an important role in the theory of infinite groups with generalized solubility conditions (see [36]). It is easy to find abelian groups to show that all these classes are distinct.

- A group G has *min* (*max*) if it satisfies the minimum (maximum) condition on subgroups.
- G is a *minimax group* if it has a finite series whose factors satisfy the minimum or maximum condition on subgroups.
- G has *finite Prüfer rank* $r = r(G)$ if every finitely generated subgroup of G can be generated by r elements and r is the least such number.
- G has *finite abelian section rank* if every elementary abelian p-section of G is finite for every prime p.
- G has *finite torsion-free rank* if it has a series of finite length whose factors are periodic or infinite cyclic. The number of infinite cyclic factors in any such series is an invariant of G called its *torsion-free rank* $r_0(G)$.

In particular a group G is *periodic* if and only if it has torsion-free rank $r_0(G) = 0$. If N is a normal subgroup of the group G with finite torsion-f ree rank, then it is easy to see that $r_0(G) = r_0(N) + r_0(G/N)$. Thus the torsion-free rank is an excellent tool for induction arguments.

3.2 The adjoint group of a radical ring

Since the Jacobson radical of a ring with minimal condition on left or right ideals is nilpotent, the adjoint group of every finite radical ring is nilpotent. The following theorem gives some information on the structure of the adjoint group of an arbitrary radical ring. Recall that a group G is an *SN-group* if it has a (general) series with abelian factors (see [37], p. 365). If $\mathfrak{X}$ is a group-theoretical property, then a group G is a *locally $\mathfrak{X}$-group*, if every finitely generated subgroup of G has the property $\mathfrak{X}$.

Theorem 3.1 (Amberg, Dickenschied and Sysak [4]) *The adjoint group R° of every radical ring R is an SN-group in which every (locally) finite subgroup is (locally) nilpotent.*

Using similar arguments as in [4], it can even be proved that in every periodic subgroup G of the adjoint group R° of a radical ring R each two elements of coprime orders are permutable. However, we do not know at present whether such a periodic subgroup is the direct product of its primary components. This is in fact the case if the annihilator of R is trivial or R° has no central Prüfer subgroups. The matter in question can be reduced to the following more general problem which is of independent interest. *Does every central extension G of a Prüfer p-group by a periodic p'-group split if G is an SN-group?* It should be noted that if G is not an SN-group, then an example of Adjan ([1], p. 276, VII.1.9) shows that there exists such a non-split extension.

Since SN-groups which satisfy *min* are locally finite (see [36], vol. 1, p. 71, Corollary), Theorem 3.1 also implies that every subgroup with *min* of the adjoint group of a radical ring is also locally nilpotent. However, it follows from a result of Neroslavskii that subgroups of the adjoint group of a radical ring which have *max* need not be locally nilpotent. In fact, in [32] a radical algebra over the finite field $GF(p)$ for some prime $p \neq 2$ is given in which the adjoint group R° contains a subgroup $G \simeq \langle a \rangle \ltimes \langle b \rangle$ with $a^{-1}ba = b^2$ and where a has infinite order and b has order p. In particular G is polycyclic, but not nilpotent. This example also shows that there exist radical rings whose adjoint group does not have a central series.

It should be noted that if a polycyclic subgroup G of the adjoint group of a radical ring R generates R as a ring, then R must be a nilpotent ring and so G is a nilpotent group. This is a consequence of a deep result of Roseblade (see [39], Theorem B). Moreover, if G is locally polycyclic with finite torsion-free rank, then R is a locally nilpotent ring and so G is a locally nilpotent group, as follows from an observation by Brown (see the remarks preceding Corollary 4.6 in [17]). It would be interesting to know *whether the last statement can be extended to the case when the group G is (locally) minimax.* A number of locally nilpotent subgroups of the adjoint group of certain radical rings are also pointed out in [4]. In particular, the adjoint group of a nil ring contains many locally nilpotent subgroups, some of which can be found by the following result.

Let $\mathfrak{X}$ be a group-theoretical property such that every $\mathfrak{X}$-group has an ascending series with locally finite or locally nilpotent factors.

Theorem 3.2 (Amberg, Dickenschied and Sysak [4]) *Let G be a subgroup of the adjoint group of a nil ring R. Then G is locally nilpotent if one of the following conditions holds:*

(i) *G is a locally $\mathfrak{X}$-group,*

(ii) *Every finitely generated subgroup of G has finite Prüfer rank.*

Moreover, if G is locally nilpotent, then the subring of R generated by G is a locally nilpotent ring.

In particular, every maximal locally nilpotent subgroup of the adjoint group of a nil ring is in fact a subring of this ring.

Observe also that examples of simple radical rings with $R^2 = R$ exist (see [42]), whereas *examples of simple nil rings are unknown.*

3.3 Finiteness condition on the adjoint group

It is interesting to investigate the relation between the adjoint group and the additive group of a radical ring R. One of the first results in this direction was obtained by Watters [51], who showed that the adjoint group of a radical ring R has *max* if and only if its additive group is finitely generated, and in this case R is nilpotent. The following theorem extends this result to the class of minimax groups.

Theorem 3.3 (Amberg and Dickenschied [3]) *Let $\mathfrak{X}$ be a class of minimax groups which is closed under the forming of subgroups, epimorphic images and extensions. The following conditions of the radical ring R are equivalent:*

(1) *The additive group R^+ is an $\mathfrak{X}$-group,*

(2) *The associated group $G(R)$ is an $\mathfrak{X}$-group,*

(3) *The adjoint group R° is an $\mathfrak{X}$-group.*

In this case R is a nilpotent ring.

Consider again the radical ring W_2 of all rational numbers with odd denominator and even numerator from Example 2.2. Clearly the additive group of W_2 has Prüfer rank 1, but its adjoint group has infinite torsion-free rank, since $1 + W_2$ is generated by -1 and all odd primes. Therefore there is no complete analogue of Theorem 3.3 for the other finiteness conditions mentioned above. However, the following theorem shows that these finiteness conditions are inherited from the adjoint group of a radical ring R to its additive group and that they imply some nilpotency conditions of the ring R.

Theorem 3.4 (Amberg and Dickenschied [3]) *Let R be a radical ring. Then the following holds:*

(a) *If R° has finite torsion-free rank n, then also $r_0(R^+) = n$, and R is a nil ring,*

(b) *If R° has finite abelian section rank, then so does R^+, and R is a T-nilpotent ring of class $cl(R) \leq \omega + r_0(R^+)$,*

(c) *If R° has finite Prüfer rank, then so does R^+, and $r(R^+)$ is bounded by a function which only depends on $r(R^\circ)$.*

Here a ring R is called *T-nilpotent of class $cl(R) = \alpha$* if $B_\alpha(R) = R$ and α is the least ordinal with this property, where the *transfinite annihilator series* of a ring R is defined by

$$B_0(R) = 0, \qquad B_{\alpha+1}(R) = \{a \in R \mid aR + Ra \subseteq B_\alpha(R)\}$$

for each ordinal α and for each limit ordinal λ

$$B_\lambda(R) = \bigcup_{\beta<\lambda} B_\alpha(R).$$

It is reasonable to ask *which is the best bound for the Prüfer rank in Theorem 3.4(c).* Note that this is unknown even if R is commutative. *Is perhaps $r(R^+) \leq 2r(R^\circ)$ in this latter case?*

The following result shows that the converse of Theorem 3.4 holds for nil rings. It is easy to see that the adjoint group of a nil ring is a p-group (torsion-free) if and only if, its additive group is a p-group (torsion-free).

Theorem 3.5 (Dickenschied [20]) *If R is a nil-ring, then the following holds:*

(a) *If R^+ has finite torsion-free rank n, then also $r_0(R^\circ) = n$,*

(b) *If R^+ has finite abelian section rank, then so does R°,*

(c) *If R^+ has finite Prüfer rank, then so does R°, and $r(R^\circ) \leq 3r(R^+)$. If R^+ contains no element of order 2 then even $r(R^\circ) \leq 2r(R^+)$.*

Statements (b) and (c) also hold for radical rings whose additive group is periodic (see [20]). On the other hand, if F is the field with p elements and $F[[x]]$ is the ring of formal power series in an indeterminate x over F, then the Jacobson radical $xF[[x]]$ of $F[[x]]$ gives an example of a radical ring R, whose additive group is an elementary abelian p-group, but whose adjoint group is torsion-free and has infinite torsion-free rank.

Which groups can occur as the adjoint group of a radical ring? This problem was in part discussed in [30]. Obviously, every abelian group A is the adjoint group of the null ring on A. For finite groups it suffices to consider only p-groups. Finite abelian groups which occur as the adjoint group of some commutative nilpotent p-algebra were discussed by Eggert in [23]. He proved that a finite abelian p-group A is the adjoint group of such an algebra if $r(A^{p^n}) \geq p \cdot r(A^{p^{n+1}})$ for every n with $A^{p^n} \neq 1$ and conjectured that the converse of this statement is also true. An affirmative answer to this conjecture would imply that the above-mentioned inequality $r(R^+) \leq 2r(R^\circ)$ holds for every commutative radical ring R.

Kaloujnine has shown in [28] that every p-group of class 2 for every odd prime p is isomorphic to the adjoint group of some nilpotent ring. In fact, all groups of order p, p^2 and p^3 occur as the adjoint of some nilpotent ring, but a group of order