I. NORMED LINEAR SPACE STRUCTURE AND EXAMPLES

Everyday Euclidean space which underlies most of our mathematical activity, possesses some remarkable abstract structures which provide a rewarding study in their own right. In particular we could isolate the linear space structure and the metric space structure and the one which is a significantly rich mixture of these two, the normed linear space structure. To study such an abstract structure is profitable for two reasons: the study can be undertaken in a systematic and deductive fashion and there are many diverse mathematical situations which exhibit the same structure to which the developed theory can be applied.

Moreover, the study of normed linear spaces has an intriguing interest which is quite distinct from that of metric spaces. Just as the study of Euclidean space gives rise to matrix theory so the linear structure of normed linear spaces enables us to propagate more normed linear spaces from the continuous linear mappings between them and this is the origin of operator theory.

§1. BASIC PROPERTIES OF NORMED LINEAR SPACES

We begin with a review of the defining structure of normed linear spaces, of the fundamental properties of continuous linear mappings and of the notions of basis in normed linear spaces. Our theory is developed from a knowledge of these and we will use them in our discussion of examples and our subsequent construction of associated normed linear spaces in Chapter II.

1.1 **Definition.** Given a linear space X over $\mathbb{C}$ (or $\mathbb{R}$), a mapping $\|\cdot\|: X \to \mathbb{R}$ is a *norm* for X if it satisfies the following properties:
For all $x \in X$,
(i) $\|x\| \geq 0$,
(ii) $\|x\| = 0$ if and only if $x = 0$,
(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all scalar λ,
and for all $x, y \in X$,
(iv) $\|x+y\| \leq \|x\| + \|y\|$.

(The norm is said to assign a "length" to each vector in X.) The pair $(X, \|\cdot\|)$ is called a *normed linear space*. Different norms can be assigned to the same linear space giving rise to different normed linear spaces.

1.2 **Remarks.** The norm generates a special metric on the linear space. Given a normed linear space $(X, \|\cdot\|)$, a function $d: X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \| x-y \|$$

is a metric for X and is called the *metric generated by the norm.*
This metric is an *invariant metric* for X; that is, for any given $z \in X$,

$$d(x+z, y+z) = d(x,y) \qquad \text{for all } x,y \in X.$$

So given $x \in X$ and $r > 0$, the ball

$$B(x; r) \equiv \{y \in X : \| y-x \| < r\} = x + \{z \in X : \| z \| < r\} \equiv x + B(0; r),$$

is the translate by x of the ball $B(0; r)$.
Furthermore, property (iii) implies that given $r > 0$, the ball centred at the origin with radius r,

$$B(0; r) = r\{y \in X : \| y \| < 1\} \equiv r\,B(0; 1),$$

is an r–multiple of the unit ball.
Properties (iii) and (iv) tell us that the norm is a *convex function* on X; that is,
for any $x,y \in X$ we have $\| \lambda x+(1-\lambda)y \| \le \lambda \| x \| + (1-\lambda) \| y \|$ for all $0 \le \lambda \le 1$.
This implies that the ball $B(0; 1)$ is a *convex set* in X; that is,
for any $x,y \in B(0; 1)$ we have $\lambda x + (1-\lambda)y \in B(0; 1)$ for all $0 \le \lambda \le 1$.
But (iii) also tells us that the ball $B(0; 1)$ is *symmetric*; that is,
for any $x \in B(0; 1)$, we have $-x \in B(0; 1)$. □

Since a normed linear space has both algebra and analysis structures, the relationship between these two quite different aspects of the space is a matter of special interest. The fruitfulness of the relationship follows from the way the linear space operations are linked to the norm.

1.3 **Remark**. Given a normed linear space $(X, \|\cdot\|)$, from properties (iii) and (iv) we can derive the important norm inequality

$$\big| \| x \| - \| y \| \big| \le \| x-y \| \qquad \text{for all } x,y \in X$$

which implies that the norm is a continuous function on X.
Properties (iii) and (iv) actually relate addition of vectors and multiplication of a vector by a scalar to the norm and imply that these algebraic operations have a continuity property which we call *joint continuity* :
if $x \to x_0$ and $y \to y_0$ then $x + y \to x_0 + y_0$, and if $\lambda \to \lambda_0$ then $\lambda x \to \lambda_0 x_0$.
This can be deduced simply from the inequalities:

$$\| (x+y) - (x_0+y_0) \| \le \| x-x_0 \| + \| y-y_0 \|,$$

and $\| \lambda x-\lambda_0 x_0 \| \le \| \lambda x-\lambda x_0 \| + \| \lambda x_0-\lambda_0 x_0 \| \le | \lambda | \| x-x_0 \| + | \lambda-\lambda_0 | \| x_0 \|$. □

A normed linear space is a generalisation of Euclidean space.

1.4 Example. *Euclidean n-space and Unitary n-space.* The real (complex) linear space of ordered n-tuples of real numbers $\mathbb{R}^n$ (of complex numbers $\mathbb{C}^n$), where for $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n)$ we define the norm

$$\|x\|_2 = \sqrt{\sum_{k=1}^{n} |\lambda_k|^2}$$

in the real case is called *Euclidean n-space* and is denoted by $(\mathbb{R}^n, \|\cdot\|_2)$ and in the complex case is called *Unitary n-space* and is denoted by $(\mathbb{C}^n, \|\cdot\|_2)$. The n-dimensional complex linear space $\mathbb{C}^n$ is a 2n-dimensional real linear space $\mathbb{C}^n(\mathbb{R})$ and it is clear that the norm calculation is not affected by the space being considered as over $\mathbb{R}$ or $\mathbb{C}$. An elegant proof that the norm satisfies the triangle inequality (property (iv)) follows from an exploration of the inner product structure of the space; (see Example 2.2.10 below). □

Other normed linear spaces are formed by taking different norms on the same underlying linear space $\mathbb{R}^n$ (or $\mathbb{C}^n$). The advantage of these norms which at first offend our Euclidean intuition, is that they are often more convenient for computation.

1.5 Examples.

(i) $(\mathbb{R}^n, \|\cdot\|_\infty)$, (or $(\mathbb{C}^n, \|\cdot\|_\infty)$).

For $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$) we define the norm

$$\| x \|_\infty = \max\{ |\lambda_k| : k \in \{1, 2, \ldots, n\}\}.$$

(ii) $(\mathbb{R}^n, \|\cdot\|_1)$, (or $(\mathbb{C}^n, \|\cdot\|_1)$).

For $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$) we define the norm

$$\| x \|_1 = \sum_{k=1}^{n} |\lambda_k|.$$

In both cases it is much simpler than in the Euclidean case to verify that these norms satisfy the norm properties (i)–(iv). □

Several example spaces can be considered as different forms of a general function space.

1.6 Examples. For any nonempty set X, the set $\mathfrak{B}(X)$ of bounded real (complex) functions on X is a linear space under pointwise definition of the linear operations and is a normed linear space with norm defined by

$$\| f \|_\infty = \sup \{| f(x) | : x \in X\}.$$

(i) When $X \equiv \{1, 2, \ldots, n\}$ then $\mathfrak{B}(X)$ is the linear space $\mathbb{R}^n$(or $\mathbb{C}^n$) and for $x \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$),

$$\| x \|_\infty = \max \{| \lambda_k | : k \in \{1, 2, \ldots, n\}\}.$$

(ii) When $X \equiv \mathbb{N}$ the set of natural numbers, then $\mathcal{B}(\mathbb{N})$ is the linear space m (or ℓ_∞) of bounded sequences of scalars and for $x \equiv \{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\} \in m$,

$$\|x\|_\infty = \sup\{|\lambda_k| : k \in \mathbb{N}\}.$$

(iii) When $X \equiv [a,b]$ the bounded closed interval, then $\mathcal{B}[a,b]$ is the linear space of bounded scalar functions on $[a,b]$ and for $f \in \mathcal{B}[a,b]$,

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}. \qquad \square$$

1.7 **Definition.** A normed linear space which is complete as a metric space with its metric generated by the norm, is called a *Banach space.*

Using the completeness of the scalar field we establish the completeness of the general example given in Example 1.6. This also illustrates the general method used to prove completeness in other examples.

1.8 **Theorem**. *For any nonempty set* X, *the normed linear space* $(\mathcal{B}(X), \|\cdot\|_\infty)$ *is a Banach space.*

Proof. Consider a Cauchy sequence $\{f_n\}$ in $(\mathcal{B}(X), \|\cdot\|_\infty)$; then given $\varepsilon > 0$ there exists a $\nu \in \mathbb{N}$ such that

$$\|f_n - f_m\|_\infty < \varepsilon \quad \text{for all } m,n > \nu;$$

that is,
$$\sup\{|(f_n - f_m)(x)| : x \in X\} < \varepsilon \quad \text{for all } m,n > \nu.$$

But then for each $x \in X$,

$$|f_n(x) - f_m(x)| < \varepsilon \quad \text{for all } m,n > \nu;$$

that is, for each $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence of scalars. Since the scalar field is complete, for each $x \in X$ we can define a function f on X by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

We need to show that $f \in \mathcal{B}(X)$.
Since $\{f_n\}$ is Cauchy it is bounded in $(\mathcal{B}(X), \|\cdot\|_\infty)$; that is, there exists a $K > 0$ such that

$$\|f_n\|_\infty < K \quad \text{for all } n \in \mathbb{N},$$

which implies that
$$|f_n(x)| < K \quad \text{for all } x \in X \text{ and all } n \in \mathbb{N}.$$

Therefore, $|f(x)| \le K$ for all $x \in X$, and so $f \in \mathcal{B}(X)$.
Then we need to show that $\{f_n\}$ actually converges to f in $(\mathcal{B}(X), \|\cdot\|_\infty)$.
We had for each $x \in X$,

$$|f_n(x) - f_m(x)| < \varepsilon \quad \text{for all } m,n > \nu.$$

Fix n and let $m \to \infty$; then we have for each $x \in X$,

$$|f_n(x) - f(x)| \le \varepsilon \quad \text{for all } n > \nu.$$

So
$$\|f_n - f\|_\infty = \sup\{|f_n(x) - f(x)| : x \in X\} \le \varepsilon \quad \text{for all } n > \nu;$$

that is, $\{f_n\}$ converges to f in $(\mathcal{B}(X), \|\cdot\|_\infty)$. $\square$

1.9 **Remarks**. So the normed linear spaces $(\mathbb{R}^n, \|\cdot\|_\infty)$, or $(\mathbb{C}^n, \|\cdot\|_\infty)$, $(m, \|\cdot\|_\infty)$ and $(\mathfrak{B}[a,b], \|\cdot\|_\infty)$ are Banach spaces. Of course, Euclidean n-space $(\mathbb{R}^n, \|\cdot\|_2)$ and Unitary n-space $(\mathbb{C}^n, \|\cdot\|_2)$ are both complete, but a proof of this will be deduced from the completeness of $(\mathbb{R}^n, \|\cdot\|_\infty)$, (or $(\mathbb{C}^n, \|\cdot\|_\infty)$ by Corollary 2.1.5. □

We will see, as we develop the fundamental theorems in our theory, that completeness plays a strategically important role.

A linear subspace in a normed linear space derives norm structure from the parent space in a natural way.

1.10 **Definition**. Given a normed linear space $(X, \|\cdot\|)$ and a linear subspace Y of X, it is clear that the restriction of the norm $\|\cdot\|$ to Y is also a norm for Y. The restriction is denoted $\|\cdot\|_Y$ and $(Y, \|\cdot\|_Y)$ is a *normed linear subspace* of $(X, \|\cdot\|)$.

The following examples are significant linear subspaces of the example spaces so far introduced.

1.11 **Examples**. In m the linear space of bounded sequences, where for each
$x \equiv \{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$ there exists an $M_x > 0$ such that $|\lambda_n| \le M_x$ for all $n \in \mathbb{N}$,
we have the following linear subspaces.
c the linear subspace of convergent sequences, where for each
$x \equiv \{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$ there exists a scalar λ such that $\lambda_n \to \lambda$ as $n \to \infty$,
c_0 the linear subspace of sequences which converge to zero,
ℓ_2 the linear subspace of sequences $x \equiv \{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$ where $\sum |\lambda_n|^2 < \infty$,
ℓ_1 the linear subspace of sequences $x \equiv \{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$ where $\sum |\lambda_n| < \infty$, and
E_0 the linear subspace of sequences with only a finite number of nonzero entries, where for each $x \equiv \{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$, $\lambda_n = 0$ for all except a finite number of elements in $\mathbb{N}$.
That ℓ_2 and ℓ_1 are linear subspaces will be verified in Example 2.2.11 and 2.3.6.
Now all of these are normed linear subspaces of $(m, \|\cdot\|_\infty)$. □

1.12 **Examples**. For a metric space (X, d), consider $\mathfrak{B}(X, d)$ the linear space of bounded functions on (X, d).
We have the important linear subspace $\mathfrak{BC}(X, d)$ of bounded continuous functions on (X, d). When (X, d) is compact, then all continuous functions on (X, d) are bounded so $\mathfrak{BC}(X, d) = \mathfrak{C}(X, d)$ the linear subspace of continuous functions on (X, d).
When $X \equiv [a,b]$, we could also consider the linear subspace $\mathfrak{R}[a,b]$ of $\mathfrak{B}[a,b]$ which consists of the Riemann integrable functions on $[a,b]$. Of course, $\mathfrak{C}[a,b]$ is a linear subspace of $\mathfrak{R}[a,b]$.

We could consider the linear subspace $\mathfrak{C}^1[a,b]$ of $\mathfrak{C}[a,b]$ which consists of those functions with a continuous first derivative on [a,b] and $\mathfrak{C}^\infty[a,b]$ which consists of functions which are infinitely often differentiable on [a,b].
We could also form linear subspaces of $\mathfrak{B}(X, d)$ of the form

$$\{f \in \mathfrak{B}(X, d) : f(x_0) = 0\} \text{ for a given } x_0 \in X.$$

So we would have $\mathfrak{B}_0[0,1]$ the linear subspace $\{f \in \mathfrak{B}[0,1] : f(0) = 0\}$.
We would also have $\mathfrak{C}_0[0,1]$ the linear subspace $\{f \in \mathfrak{C}[0,1] : f(0) = 0\}$.
Again all of these are normed linear subspaces of the appropriate bounded function space with the supremum norm $\|\cdot\|_\infty$. □

Often the simplest way to examine a normed linear space for completeness is to use the following metric space link between completeness and closedness, (see AMS §4).

1.13 **Proposition**. *Consider a metric space* (X, d) *and a subset* Y.
(i) *If* $(Y, d|_Y)$ *is complete then* Y *is a closed subset of* (X, d).
(ii) *If* (X, d) *is complete and* Y *is a closed subset of* (X, d) *then* $(Y, d|_Y)$ *is complete.*

1.14 **Example**. Given a metric space (X, d), the normed linear space $(\mathfrak{B}\mathfrak{C}(X, d), \|\cdot\|_\infty)$ is complete.

Proof. We show that $\mathfrak{B}\mathfrak{C}(X, d)$ is a closed subset of $(\mathfrak{B}(X, d), \|\cdot\|_\infty)$ and apply Proposition 1.13.
Given a cluster point f of $\mathfrak{B}\mathfrak{C}(X, d)$ in $(\mathfrak{B}(X, d), \|\cdot\|_\infty)$ we show that f is continuous:
Now there exists a sequence $\{f_n\}$ in $\mathfrak{B}\mathfrak{C}(X, d)$ convergent to f ; that is, given $\varepsilon > 0$ there exists a $\nu \in \mathbb{N}$ such that

$$\|f_n - f\|_\infty < \varepsilon \quad \text{when } n > \nu.$$

Consider $x_0 \in X$. Since $f_{\nu+1}$ is continuous at x_0 there exists a $\delta > 0$ such that

$$|f_{\nu+1}(x) - f_{\nu+1}(x_0)| < \varepsilon \quad \text{when } d(x,x_0) < \delta.$$

Therefore,

$$\begin{aligned} |f(x)-f(x_0)| &\le |f(x) - f_{\nu+1}(x)| + |f_{\nu+1}(x) - f_{\nu+1}(x_0)| + |f_{\nu+1}(x_0) - f(x_0)| \\ &\le 2\|f_{\nu+1} - f\|_\infty + |f_{\nu+1}(x) - f_{\nu+1}(x_0)| \\ &< 3\varepsilon \qquad \text{when } d(x,x_0) < \delta; \end{aligned}$$

that is, f is continuous at x_0. □

When (X, d) is compact, the Banach space $\mathfrak{C}(X, d)$ has a particularly rich structure, (see AMS §9).

We often encounter situations where we have a linear space with a real function which has all the norm properties except (ii).

1.15 **Definition**. Given a linear space X over $\mathbb{C}$ (or $\mathbb{R}$), a mapping $p: X \to \mathbb{R}$ is a *seminorm* for X if it satisfies all the norm properties except (ii) and instead of (ii) satisfies

(ii)' $\quad p(x) = 0$ if $x = 0$.

The pair (X,p) is called a *seminormed linear space*.

So a seminorm allows the possibility that there exists some $x \neq 0$ for which $p(x) = 0$.

1.16 **Remark**. A seminorm p on a linear space X generates a semimetric e on X defined by

$$e(x,y) = p(x-y).$$

As with the norm properties noted in Remarks 1.2, property (iii) implies that the set $\{x \in X : p(x) < 1\}$ has the properties that, given $r > 0$

$$\{y \in X : p(y-x) < r\} = x + r\{z \in X : p(z) < 1\}$$

and is convex and symmetric.

However, if p is not a norm then ker p is a nontrivial linear subspace and is a subset of $\{x \in X : p(x) < 1\}$. □

Given a linear space and a proper linear subspace, another linear space can be generated in a natural way as a quotient space.

1.17 **Definition.** Given a linear space X and a proper linear subspace M of X, the *quotient space* (or *factor space*) X/M is the linear space of *cosets* $[x] \equiv x + M$, with addition and multiplication by a scalar defined by

$$[x] + [y] \equiv (x + M) + (y + M) = (x + y) + M \equiv [x+y] \quad \text{for all } [x],[y] \in X/M$$

and $$\lambda[x] \equiv \lambda(x+M) = \lambda x + M \equiv [\lambda x] \quad \text{for all } [x] \in X/M \text{ and scalar } \lambda.$$

A seminormed linear space can be transformed into a normed linear space as a quotient space.

1.18 **Definition**. Given a seminormed linear space (X, p), the *associated normed linear space* is the quotient space X/ker p whose elements are the cosets $[x] \equiv x + \ker p$, with norm

$$\| [x] \| = p(x) \qquad \text{for any } x \in [x].$$

1.19 **Example**. Consider $\mathcal{R}[a,b]$ the linear space of Riemann integrable functions on [a,b]. Now the function $p_1: \mathcal{R}[a,b] \to \mathbb{R}$ defined by

$$p_1(f) = \int_a^b | f(t) | \, dt$$

is a seminorm on $\mathcal{R}[a,b]$.

On the quotient space $\mathcal{R}[a,b]/\ker p_1$ of cosets $[f] = f+\ker p_1$, we have the associated norm

$$\| [f] \|_1 = p_1(f) \quad \text{for any } f \in [f].$$ □

When a quotient space is generated from a normed linear space by a proper closed linear subspace then the quotient space has an associated quotient norm.

1.20 **Definition.** Given a normed linear space $(X, \|\cdot\|)$, and a proper closed linear subspace M of $(X, \|\cdot\|)$, the *quotient norm* $\|\cdot\|$ is defined on the quotient space X/M by

$$\| [x] \| = d(0, x+M) = \inf\{ \| x+m \| : m \in M\}.$$

1.21 **Remark.** It is routine to verify the norm properties (i)–(iv) for the quotient norm. But we note that we need the linear subspace M to be closed to establish that the norm is not just a seminorm; that is, we need to show that $d(0, x+M) = 0$ implies that $x + M = M$ so that $\| [x] \| = 0$ implies that $[x] = 0$. □

It is instructive to see how such a quotient space inherits completeness from the parent space.

1.22 **Theorem.** *Given a Banach space* $(X, \|\cdot\|)$ *and a proper closed linear subspace* M *then the quotient space* $(X/M, \|\cdot\|)$ *is also complete.*

Proof. Consider a Cauchy sequence $\{x_n+M\}$ in $(X/M, \|\cdot\|)$. Then for each $k \in \mathbb{N}$ there exists a $\nu(k) \in \mathbb{N}$ such that

$$\| (x_n + M) - (x_m + M) \| < \frac{1}{2^k} \quad \text{for all } m,n > \nu(k).$$

Consider a subsequence of the form $\{x_{n(k)} + M\}$ where for each $k \in \mathbb{N}$, $n(k) > \nu(k)$.

Then $$\| (x_{n(k)} + M) - (x_{n(k+1)} + M) \| < \frac{1}{2^k}.$$

For each $k \in \mathbb{N}$ choose $x_k \in x_{n(k)} + M$ such that

$$\| x_k - x_{k+1} \| < \frac{1}{2^k}.$$

The sequence $\{x_k\}$ in $(X, \|\cdot\|)$ has the property that

$$\begin{aligned} \| x_k - x_m \| &\le \| x_k - x_{k+1} \| + \dots + \| x_{m-1} - x_m \| \quad \text{for } m > k \\ &< \frac{1}{2^{k-1}} \quad \text{for all } k \in \mathbb{N}, \end{aligned}$$

so $\{x_k\}$ is a Cauchy sequence in $(X, \|\cdot\|)$. But $(X, \|\cdot\|)$ is complete so there exists an $x \in X$ such that $\{x_k\}$ is convergent to x. Then

$$\| (x_{n(k)}+M) - (x+M) \| = \| (x_k+M) - (x+M) \| \le \| x_k - x \|$$

and so $\{x_{n(k)}+M\}$ is convergent to $x + M$ in $(X/M, \|\cdot\|)$.
However, $\{x_{n(k)}+M\}$ is a convergent subsequence of the original Cauchy sequence $\{x_n+M\}$, so $\{x_n+M\}$ is also convergent in $(X/M, \|\cdot\|)$. □

1.23 **Example**. Consider $\mathcal{R}[a,b]$ the linear space of Riemann integrable functions on $[a,b]$ with norm

$$\| f \|_\infty = \sup\{ | f(t) | : t \in [a,b] \}.$$

Now $(\mathcal{R}[a,b], \|\cdot\|_\infty)$ is a Banach space, (see AMS §4).

Furthermore, ker p_1 where

$$p_1(f) = \int_a^b | f(t) | \, dt$$

is a closed linear subspace of $(\mathcal{R}[a,b], \|\cdot\|_\infty)$. On the quotient space $\mathcal{R}[a,b]/\ker p_1$ whose elements are cosets, $[f] = f + \ker p_1$, we have the norm

$$\| [f] \|_\infty = d(0, f+\ker p_1) = \inf\{ \| f+g \|_\infty : g \in \ker p_1 \} .$$

From Theorem 1.22, the quotient space $(\mathcal{R}[a,b]/\ker p_1, \|\cdot\|_\infty)$ is a Banach space. □

1.24 Continuous linear mappings

The algebraic study of linear spaces finds its full development in an examination of the homomorphisms or structure preserving mappings between such spaces. As we might expect, these linear mappings are of similar significance in the development of the analysis of normed linear spaces. However, interest in this case is focused on the continuous linear mappings which preserve the topological and norm structure along with the linear structure.

The following characterisation theorem provides an essential tool for discussing these mappings.

1.24.1 **Theorem**. *Given normed linear spaces* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|')$, *a linear mapping* $T: X \to Y$

(i) *is continuous if and only if there exists an* $M > 0$ *such that*

$$\| Tx \|' \le M \| x \| \quad \textit{for all } x \in X,$$

(ii) *has a continuous inverse on* $T(X)$ *if and only if there exists an* $m > 0$ *such that*

$$m \| x \| \le \| Tx \|' \quad \textit{for all } x \in X.$$

Proof.

(i) If the condition holds then clearly T is continuous at 0 and the linearity of T implies that T is continuous on X.

Conversely, suppose that the condition does not hold; that is, for each $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$\| Tx_n \|' > n \| x_n \|.$$

Then $$\left\| \frac{1}{n} \frac{x_n}{\| x_n \|} \right\| = \frac{1}{n} \to 0 \text{ as } n \to \infty,$$

but $$\left\| T\left(\frac{1}{n} \frac{x_n}{\| x_n \|}\right) \right\| > 1 \quad \text{for all } n \in \mathbb{N};$$

that is, T is not continuous at 0.

(ii) If there exists an $m > 0$ such that

$$m \| x \| \le \| Tx \|' \qquad \text{for all } x \in X.$$

then $\ker T = \{0\}$ so T is one-to-one and T^{-1} exists on $T(X)$ and is clearly linear. Writing $x = T^{-1}y$ for $y \in T(X)$ we have

$$m \| T^{-1}y \| \le \| T(T^{-1}y) \|' = \| y \|'$$

so $$\| T^{-1} y \| \le \frac{1}{m} \| y \|' \qquad \text{for all } y \in T(X);$$

that is, from (i) T^{-1} is continuous on $T(X)$.

Conversely, if T^{-1} is a continuous linear mapping on $T(X)$ then, from (i) there exists an $M > 0$ such that

$$\| T^{-1}y \| \le M \| y \| \qquad \text{for all } y \in T(X).$$

Since $y = Tx$, $$\| T^{-1}(Tx) \| \le M \| Tx \|'$$

and so $$\frac{1}{M} \| x \| \le \| Tx \|' \qquad \text{for all } x \in X.$$ □

1.24.2 **Remark.** From linearity it follows that a linear mapping T is continuous on a normed linear space $(X, \|\cdot\|)$ if and only if T is continuous at any one point of X. It follows that a linear mapping T is either continuous at every point of X or continuous at no point of X. □

Two of the most commonly occurring linear mappings are as follows.

1.24.3 **Examples.**
Consider the normed linear spaces $(\mathfrak{C}[0,1], \|\cdot\|_\infty)$ and $(\mathfrak{C}^1[0,1], \|\cdot\|_\infty)$.

(i) Consider the linear mapping $I: \mathfrak{C}[0,1] \to \mathfrak{C}^1[0,1]$ defined by

$$I(f)(x) = \int_0^x f(t)\, dt \qquad \text{for all } x \in [0,1].$$

Now $$| I(f)(x) | \le | x | \| f \|_\infty \qquad \text{for all } x \in [0,1]$$

and $$\| I(f) \|_\infty = \max\{ | I(f)(x) | : x \in [0,1] \} \le \| f \|_\infty \text{ for all } f \in \mathfrak{C}[0,1],$$

so I is continuous.

(ii) Consider the linear mapping $D: \mathfrak{C}^1[0,1] \to \mathfrak{C}[0,1]$ defined by

$$D(f)(x) = f'(x) \text{ for all } x \in [0,1].$$

For the sequence $\{f_n\}$ in $\mathfrak{C}^1[0,1]$ where $f_n(x) = \frac{1}{n} \sin n\pi x$ we have $\| f_n \|_\infty = \frac{1}{n} \to 0$ but $\| D(f_n) \|_\infty \doteq 1$ for all $n \in \mathbb{N}$, so D is not continuous. □

1.24.4 **Definitions.** Consider normed linear spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ and a linear mapping $T: X \to Y$.

(i) T is said to be a *topological isomorphism* (or a *linear homeomorphism*) if T is also a homeomorphism; that is, T is linear, continuous, invertible and has a continuous inverse on $T(X)$.