Fluctuations in Physical Systems

This book provides an introduction to applied statistical mechanics by considering physically realistic models and examples. It gives a simple and accessible introduction to theories of thermal fluctuations and diffusion, and goes on to apply them in a variety of physical contexts.

The first part of the book is devoted to processes in thermal equilibrium, and considers linear systems. Starting with an introduction to the ideas underlying the subject, the text then develops the necessary mathematical tools and applies them to fluctuations and diffusion processes in a number of contexts. Ideas central to the subject, such as the fluctuation–dissipation theorem, Fokker–Planck equations and the Kramers–Kronig relations, are introduced in a natural way during the course of the exposition. The book goes on to expand the scope by including nonequilibrium systems and also illustrates some simple nonlinear systems.

This book will be of interest to final year undergraduate and graduate students studying statistical mechanics, plasma physics, basic electronics, solid-state physics and anyone who wants an accessible introduction to the subject.

HANS PÉCSELI obtained his PhD on experimental studies of plasma waves in 1974 at the Technical University of Denmark. He was research associate at the Risø National Laboratory in Denmark (1974–1991), apart from a sabbatical year at the University of Iowa, Iowa City, in 1979–1980.

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Hans Pécseli received the annual physics prize of the Danish Physical Society for 1989 together with O. Kofoed-Hansen, for works on plasma turbulence. He is a fellow of the American Physical Society, a member of the Norwegian Academy of Sciences, and a member of the Royal Danish Academy of Sciences and Letters.

He is the author or co-author of more than 150 scientific publications on plasma physics and related topics, not to mention a similar number of conference contributions.

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The principle of generating small amounts of *finite* improbability by simply hooking the logic circuits of a Bambleweeny 57 Sub-Meson Brain to an atomic vector plotter suspended in a strong Brownian Motion producer (such as a hot cup of tea) were of course well understood – and such generators were often used to break the ice at parties by making all the molecules in the hostess's undergarments leap simultaneously one foot to the left, in accordance with the Theory of Indeterminacy.

Many respectable physicists said that they weren't going to stand for this, partly because it was a debasement of science, but mostly because they didn't get invited to those sorts of parties.

(Douglas Adams, *The Hitchhiker's Guide to the Galaxy*)

To Henriette, Maria and Thomas

Contents

Preface

These notes were originally prepared for a series of lectures on fluctuation phenomena at the Danish Space Research Institute many years ago. In their present form, they were compiled for lectures at the University of Tromsø. I should like to thank my students and colleagues for their enthusiasm and for many stimulating discussions. In particular, I should like to express my gratitude to Professor Asger Nielsen from the Technical University of Denmark who first introduced me to this subject when I followed his lectures for my Ph.D. The present summary is indebted, in content as well as form of presentation, to his inspiring lectures and lecture notes. I am greatly indebted also to Liv Larssen at the Auroral Observatory of the University of Tromsø for her tireless and expert assistance with typing and editing the manuscript, and with preparation of figures. Also the friendly advice and support of Jan Trulsen and Bård Krane are gratefully acknowledged. The text was prepared by using LATEX and some of the figures were prepared by using Mathematica.

It is hoped that no particular prior knowledge apart from that gained from basic mathematics and statistics courses will be required. Introductory courses in thermodynamics and statistical mechanics will be a great advantage, though. Since many of the examples are taken from electronics, some basic knowledge of elementary electric circuits will also be a help. The notes advocate learning by working through examples, although this was not the original intention. On the other hand, it might not be such a bad idea after all. A detailed discussion of $1/f$ noise is deliberately omitted; this is an interesting and important topic with new developments, but it was felt to be outside the scope of the present book.

There are several repetitions in the text; it is hoped that this makes it easier to read. Care is taken to ensure that the contents of the individual chapters are interrelated, at least to some degree, and that they illustrate different aspects and consequences of some basic phenomena. There are examples and exercises in the text, where many of the examples can be considered as ‘solved problems.’ More can be found in the compilations of for instance Takács (1960) and Sveshnikov (1978). In particular, problems with solutions of the diffusion equation in various geometries and with various boundary conditions are discussed in great detail by Carslaw and Jaeger (1959).

The reference list has an emphasis on the classical literature on the subject and is far from complete, but ought to serve as a useful starting point for further study. In some cases a reference may seem redundant; probably nobody will check Brown’s original paper on the motion named after him! On the other hand, it may be interesting to have the actual year and title of the publication of his discovery stated explicitly. Other papers, like Einstein’s for instance, are certainly still worthwhile reading and are therefore given with references that are readily available, rather than the originals. Even more so with the references to Rice’s papers; few have ever seen the originals in the Bell Lab. Journals, and the excellent book of reprints on statistical physics compiled by Nelson Wax is the only readily available reference for this and several other papers of fundamental importance.

An outline of individual chapters

- **Chapter 1:** This is a brief introduction containing some general discussion.
- **Chapter 2:** An outline of basics of statistical analysis is presented. The reader will, I assume, be familiar with most of the contents of this chapter, but it is considered an advantage that the text is self-contained to some degree. Moment-generating functions and characteristic functions are defined, since they will be used in later chapters. Because of their importance, the Wiener–Khinchine relations are discussed in some detail.
- **Chapter 3:** Thermal fluctuations in electric circuits are illustrated by a specific example. The model is physically realistic and deterministic, apart from a specific statistical assumption concerning a collisional process. The results illustrate the Nyquist fluctuation theorem.
- **Chapter 4:** The Nyquist fluctuation theorem, or the more general fluctuation–dissipation theorem, is derived from basic thermodynamic arguments for one specific system, and straightforward generalizations of the theorem are argued.
- **Chapter 5:** The Kramers–Kronig relations are derived and their consequences for response functions, such as dielectric functions, are discussed.
- **Chapter 6:** Brownian motion, being historically the first thoroughly described observation of thermal motion, is discussed. Simple models, as well as analytic descriptions of the phenomena, based on the Langevin equation, are presented.
- **Chapter 7:** The description of Brownian motion is completed by a simple random-walk model that can be analyzed in detail, with particular attention to the presence of reflecting or absorbing boundaries. The Lévi-flight model of a random walk is outlined.
- **Chapter 8:** Density fluctuations in gases are discussed, with emphasis on the fluctuation at a critical point.
- **Chapter 9:** A simple statistical model for random processes proposed, for instance, by Rice is discussed in detail, following his exposition. The model is used to illustrate the effect of finite record lengths on the uncertainty in estimators, for instance. Shot noise is used as an example for a practical application of the model.
- **Chapter 10:** Markov processes are discussed. Master equations are introduced, and Fokker–Planck equations derived as a special case. The chapter provides a mathematical basis for analyzing a number of stochastic or random phenomena, which is to be applied in following chapters.
- **Chapter 11:** Strictly speaking, the Fokker–Planck equation is derived for probability densities. For physical systems, an equation for particle densities in a given realization, which is not the same thing, is usually desired. In this chapter equivalents of Fokker–Planck equations are derived and applied to important problems such as sedimentation in a gravitational field and coagulation in colloids. The latter

problem is particularly interesting, in that it concerns a system that is not stationary, since the density of the macroscopic particles is changing with time.

- **Chapter 12:** Thermal fluctuations in a nonlinear circuit element are discussed, using a diode as an example. A general master equation is derived for the problem, and its solution in terms of characteristic functions demonstrated. A Fokker–Planck equation is derived as a limiting case.
- **Chapter 13:** Acceleration of light particles by random motion, Fermi acceleration, is discussed and illustrated by specific models. A master equation is derived and solved for a specific case. The analysis of the stochastic model is presented in terms of generating functions. A deterministic model is introduced and the possibility of chaotic phenomena demonstrated.
- **Appendixes:** Three of the six appendixes present basic statistical distributions: The Binomial distribution, for discrete and finite variable space, the Poisson distribution for discrete infinite variable space, and the Gaussian distribution for a continuous, infinite range of variables. Because of the frequent use of Dirac's δ-function in this book it was found advantageous to include also an appendix concerning its properties, without going into detail in discussions of generalized functions, though.

The letter T denotes 'temperature' throughout and always appears together with Boltzmann's constant κ, whereas $\mathcal{T}$ indicates a (large) time interval. There being a shortage of letters, implies that some other symbols can have different meanings in different chapters, the worst problem being that the letter i can mean electric current as well as $\sqrt{-1}$. The correct interpretation will in all cases be evident from the context.