Analysis in Integer and Fractional Dimensions

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I

A Prologue: Mostly Historical

1 From the Linear to the Bilinear

At the start and at the very foundation, there is the *Riesz representation theorem.* In original form it is

Theorem 1 (F. Riesz, 1909). *Every bounded, real-valued linear functional α on $\mathrm{C}([a,b])$ can be represented by a real-valued function g of bounded variation on $[a,b]$, such that*

$$\alpha(f) = \int_a^b f \, \mathrm{d}g, \qquad f \in \mathrm{C}([a,b]), \tag{1.1}$$

where the integral in (1.1) *is a Riemann–Stieltjes integral.*

The measure-theoretic version, headlined also the *Riesz representation theorem*, effectively marks the beginning of functional analysis. In general form, it is

Theorem 2 *Let X be a locally compact Hausdorff space. Every bounded, real-valued linear functional on $\mathrm{C}_0(X)$ can be represented by a regular Borel measure ν on X, such that*

$$\alpha(f) = \int_X f \, \mathrm{d}\nu, \qquad f \in \mathrm{C}_0(X). \tag{1.2}$$

And in its most primal form, measure-theoretic (and non-trivial!) details aside, the theorem is simply

Theorem 3 *If α is a real-valued, bounded linear functional on* $c_0(\mathbb{N}) = c_0$*, then*

$$\|\hat{\alpha}\|_1 := \sum_n |\hat{\alpha}(n)| < \infty, \tag{1.3}$$

and

$$\alpha(f) = \sum_n \hat{\alpha}(n) f(n), \quad f \in c_0,$$

where $\hat{\alpha}(n) = \alpha(\mathbf{e}_n)$ $(\mathbf{e}_n(n) = 1,$ *and* $\mathbf{e}_n(j) = 0$ *for* $j \neq n)$.

The proof of Theorem 3 is merely an observation, which we state in terms of the Rademacher functions.

Definition 4 A Rademacher system indexed by a set E is the collection $\{r_x : x \in E\}$ of functions defined on $\{-1,1\}^E$, such that for $x \in E$

$$r_x(\omega) = \omega(x), \quad \omega \in \{-1,1\}^E. \tag{1.4}$$

To obtain the first line in (1.3), note that

$$\sup\left\{\left\|\sum_{n=1}^{N} \hat{\alpha}(n)\ r_n\right\|_\infty : N \in \mathbb{N}\right\} = \|\hat{\alpha}\|_1, \tag{1.5}$$

and to obtain the second, use the fact that finitely supported functions on $\mathbb{N}$ are norm-dense in $c_0(\mathbb{N})$.

Soon after F. Riesz had established his characterization of bounded *linear* functionals, M. Fréchet succeeded in obtaining an analogous characterization in the *bilinear* case. (Fréchet announced the result in 1910, and published the details in 1915 [Fr]; Riesz's theorem had appeared in 1909 [Ri$_f$1].) The novel feature in Fréchet's characterization was a two-dimensional extension of the *total variation* in the sense of Vitali. To wit, if f is a real-valued function on $[a,b] \times [a,b]$, then the *total variation* of f can be expressed as

$$\sup\left\{\left\|\sum_{n,m} \Delta^2 f(x_n, y_m)\ r_{nm}\right\|_\infty : a < \cdots < x_n < \cdots < b,\right.$$
$$\left. a < \cdots < y_m < \cdots < b\right\}, \tag{1.6}$$

where Δ^2 is the 'second difference',

$$\begin{aligned}&\Delta^2 f(x_n, y_m)\\ &\quad = f(x_n, y_m) - f(x_{n-1}, y_m) + f(x_{n-1}, y_{m-1}) - f(x_n, y_{m-1}),\end{aligned} \tag{1.7}$$

and $\{r_{nm} : (n,m) \in \mathbb{N}^2\}$ is the Rademacher system indexed by $\mathbb{N}^2$. The two-dimensional extension of this one-dimensional measurement is given by:

Definition 5 The Fréchet variation of a real-valued function f on $[a,b] \times [a,b]$ is

$$\|f\|_{F_2} = \sup\left\{ \left\| \sum_{n,m} \Delta^2 f(x_n, y_m)\ r_n \otimes r_m \right\|_\infty : a < \cdots < x_n < \cdots < b, \right. \\ \left. < \cdots < y_m < \cdots < b \right\}. \tag{1.8}$$

($r_n \otimes r_m$ is defined on $\{-1,1\}^{\mathbb{N}} \times \{-1,1\}^{\mathbb{N}}$ by

$$r_n \otimes r_m(\omega_1, \omega_2) = \omega_1(n)\omega_2(m),$$

and $\|\cdot\|_\infty$ is the supremum over $\{-1,1\}^{\mathbb{N}} \times \{-1,1\}^{\mathbb{N}}$.)

Based on (1.8), the bilinear analog of Riesz's theorem is

Theorem 6 (Fréchet, 1915). *A real-valued bilinear functional β on* $\mathrm{C}([a,b])$ *is bounded if and only if there is a real-valued function h on $[a,b] \times [a,b]$ with $\|h\|_{F_2} < \infty$, and*

$$\beta(f,g) = \int_a^b \int_a^b f \otimes g \ \mathrm{d}h, \quad f \in \mathrm{C}([a,b]),\ g \in \mathrm{C}([a,b]), \tag{1.9}$$

where the right side of (1.9) *is an iterated Riemann–Stieltjes integral.*

The crux of Fréchet's proof was a construction of the integral in (1.9), a non-trivial task at the start of the twentieth century when integration theories had just begun developing.

Like Riesz's theorem, Fréchet's theorem can also be naturally recast in the setting of locally compact Hausdorff spaces; we shall come to this in good time. At this juncture we will prove only its primal version.

Theorem 7 *If β is a bounded bilinear functional on c_0, and $\beta(\mathbf{e}_m, \mathbf{e}_n) := \hat{\beta}(m,n)$, then*

$$\sup\left\{\left\|\sum_{m\in S, n\in T} \hat{\beta}(m,n)\ r_m \otimes r_n\right\|_\infty : \text{finite sets } S \subset \mathbb{N}, T \subset \mathbb{N}\right\}$$
$$:= \|\hat{\beta}\|_{F_2} < \infty, \tag{1.10}$$

and

$$\begin{aligned}\beta(f,g) &= \sum_{m=1}^\infty \left(\sum_{n=1}^\infty \hat{\beta}(m,n)\ g(n)\right) f(m)\\ &= \sum_{n=1}^\infty \left(\sum_{m=1}^\infty \hat{\beta}(m,n)\ f(m)\right) g(n),\\ & \qquad f \in c_0, \quad g \in c_0.\end{aligned} \tag{1.11}$$

Conversely, if $\hat{\beta}$ is a real-valued function on $\mathbb{N}\times\mathbb{N}$ such that $\|\hat{\beta}\|_{F_2} < \infty$, then (1.11) defines a bounded bilinear functional on c_0.

The key to Theorem 7 is

Lemma 8 *If $\hat{\beta} = (\hat{\beta}(m,n) : (m,n) \in \mathbb{N}^2)$ is a scalar array, then*

$$\|\hat{\beta}\|_{F_2} = \sup\left\{\left|\sum_{m\in S, n\in T} \hat{\beta}(m,n)\, x_m\, y_n\right| : x_m \in [-1,1],\right.$$
$$\left. y_n \in [-1,1], \text{ finite sets } S \subset \mathbb{N}, T \subset \mathbb{N}\right\}. \tag{1.12}$$

Proof: The right side obviously bounds $\|\hat{\beta}\|_{F_2}$. To establish the reverse inequality, suppose S and T are finite subsets of $\mathbb{N}$, and $\omega \in \{-1,1\}^{\mathbb{N}}$. Then

$$\begin{aligned}\|\hat{\beta}\|_{F_2} &\geq \left\|\sum_{n\in T, m\in S} \hat{\beta}(m,n)\ r_m \otimes r_n\right\|_\infty,\\ &\geq \sum_{n\in T}\left|\sum_{m\in S} \hat{\beta}(m,n)\ r_m(\omega)\right|.\end{aligned} \tag{1.13}$$

If $y_n \in [-1,1]$ for $n \in T$, then the right side of (1.13) bounds

$$\left|\sum_{n\in T}\left(\sum_{m\in S}\hat{\beta}(m,n)\ r_m(\omega)\right)y_n\right| = \left|\sum_{m\in S}\left(\sum_{n\in T}\hat{\beta}(m,n)\ y_n\right)r_m(\omega)\right|. \tag{1.14}$$

By maximizing the right side of (1.14) over $\omega \in \{-1,1\}^{\mathbb{N}}$, we conclude that $\|\hat{\beta}\|_{F_2}$ bounds

$$\sum_{m\in S}\left|\sum_{n\in T}\hat{\beta}(m,n)\ y_n\right|. \tag{1.15}$$

If $x_m \in [-1,1]$ for $m \in S$, then (1.15) bounds

$$\left|\sum_{m\in S}\left(\sum_{n\in T}\hat{\beta}(m,n)\ y_n\right)x_m\right|$$

$$= \left|\sum_{m\in S,n\in T}\hat{\beta}(m,n)\,x_m\,y_n\right|, \tag{1.16}$$

which implies that $\|\hat{\beta}\|_{F_2}$ bounds the right side of (1.12). □

Proof of Theorem 7: If β is a bilinear functional on c_0, with norm $\|\beta\| := \sup\{|\beta(f,g)| : f \in B_{\mathrm{c}_0},\ g \in B_{\mathrm{c}_0}\}$, then (because finitely supported functions are norm-dense in c_0)

$$\|\beta\| = \sup\left\{\left|\sum_{m\in S,n\in T}\hat{\beta}(m,n)\ x_m\ y_n\right| : x_m \in [-1,1],\right.$$

$$\left. y_n \in [-1,1],\ \text{finite sets } S \subset \mathbb{N}, T \subset \mathbb{N}\right\},$$

and Lemma 8 implies (1.10).

Let $f \in \mathrm{c}_0$ and $g \in \mathrm{c}_0$. If $N \in \mathbb{N}$, then let $f_N = f\mathbf{1}_{[N]}$ and $g_N = g\mathbf{1}_{[N]}$. (Here and throughout, $[N] = \{1,\ldots,N\}$.) Because $f_N \to f$ and $g_N \to g$ as $N \to \infty$ (convergence in c_0), and β is continuous in each coordinate, we obtain $\beta(f_N,g) \to \beta(f,g)$ and $\beta(f,g_N) \to \beta(f,g)$ as $N \to \infty$, and then obtain (1.11) by noting that $\beta(f_N,g_N) = \Sigma_{m=1}^N\Sigma_{n=1}^N\hat{\beta}(m,n)g(n)f(m)$.

Conversely, if $\hat{\beta}$ is a scalar array on $\mathbb{N}\times\mathbb{N}$, and f and g are finitely supported real-valued functions on $\mathbb{N}$, then define

$$\beta(f,g) = \sum_m\sum_n\hat{\beta}(m,n)g(n)f(m). \tag{1.17}$$

By Lemma 8 and the assumption $\|\hat{\beta}\|_{F_2} < \infty$, β is a bounded bilinear functional on a dense subspace of c_0, and therefore determines a bounded bilinear functional on c_0. The first part of the theorem implies (1.10) and (1.11).

Theorem 7 was elementary, basic, and straightforward – view it as a warm-up. In passing, observe that whereas every bounded linear functional on c_0 obviously extends to a bounded linear functional on l^∞, the analogous fact in two dimensions, that every bounded bilinear functional on c_0 extends to a bounded bilinear functional l^∞ is also elementary, but not quite as easy to verify. This 'two-dimensional' fact, specifically that (1.11) extends to f and g in l^∞, will be verified in a later chapter.

2 A Bilinear Theory

Notably, Fréchet did not consider in his 1915 paper the question whether there exist functions with bounded variation in his sense, but with infinite total variation in the sense of Vitali. Whether *bilinear* functionals on $C([a, b])$ can be distinguished from *linear* functionals on $C([a, b]^2)$ is indeed a basic and important issue (Exercises 1, 2, 4, 8). So far as I can determine, Fréchet never considered or raised it (at least, not in print). Be that as it may, this question led directly to the next advance.

Littlewood began his classic 1930 paper [Lit4] thus: 'Professor P.J. Daniell recently asked me if I could find an example of a function of two variables, of bounded variation according to a certain definition of Fréchet, but not according to the usual definition.' Noting that the problem was equivalent to finding real-valued arrays

$$\hat{\beta} = (\hat{\beta}(m, n) : (m, n) \in \mathbb{N}^2)$$

with $\|\hat{\beta}\|_{F_2} < \infty$ and $\|\hat{\beta}\|_1 = \Sigma_{m,n}|\hat{\beta}(m, n)| = \infty$, Littlewood settled the problem by a quick use of the Hilbert inequality (Exercise 1). He then considered this question: whereas there are $\hat{\beta}$ with $\|\hat{\beta}\|_{F_2} < \infty$ and $\|\hat{\beta}\|_1 = \infty$, and (at the other end) $\|\hat{\beta}\|_{F_2} < \infty$ implies $\|\hat{\beta}\|_2 < \infty$ (Exercise 3), are there $p \in (1, 2)$ such that

$$\|\hat{\beta}\|_{F_2} < \infty \Rightarrow \|\hat{\beta}\|_p < \infty?$$

Littlewood gave this precise answer.

Theorem 9 (the 4/3 inequality, 1930).

$$\|\hat{\beta}\|_p < \infty \ \textit{ for all } \ \hat{\beta} \ \textit{ with } \ \|\hat{\beta}\|_{F_2} < \infty \ \textit{ if and only if } \ p \geq \frac{4}{3}.$$

To establish 'sufficiency', that $\|\hat{\beta}\|_{F_2} < \infty$ implies $\|\hat{\beta}\|_{4/3} < \infty$, Littlewood proved and used the following:

Theorem 10 (the mixed (l^1, l^2)-norm inequality, 1930). *For all real-valued arrays $\hat{\beta} = (\hat{\beta}(m,n) : (m,n) \in \mathbb{N}^2)$,*

$$\sum_m \left(\sum_n |\hat{\beta}(m,n)|^2 \right)^{\frac{1}{2}} \leq \kappa \|\hat{\beta}\|_{F_2}, \tag{2.1}$$

where $\kappa > 0$ is a universal constant.

This mixed-norm inequality, which was at the heart of Littlewood's argument, turned out to be a precursor (if not a catalyst) to a subsequent, more general inequality of Grothendieck. We shall come to Grothendieck's inequality in a little while.

To prove 'necessity', that there exists $\hat{\beta}$ with $\|\hat{\beta}\|_{F_2} < \infty$ and

$$\|\hat{\beta}\|_p = \infty \text{ for all } p < 4/3,$$

Littlewood used the *finite Fourier transform.* (You are asked to work this out in Exercise 4, which, like Exercise 1, illustrates first steps in harmonic analysis.)

Besides motivating the inequalities we have just seen, Fréchet's 1915 paper led also to studies of 'bilinear integration', first by Clarkson and Adams in the mid-1930s (e.g., [ClA]), and then by Morse and Transue in the late 1940s through the mid-1950s (e.g., [Mor]). For their part, firmly believing that the two-dimensional framework was interesting, challenging, *and* important, Morse and Transue launched extensive investigations of what they dubbed *bimeasures*: bounded bilinear functionals on $C_0(X) \times C_0(Y)$, where X and Y are locally compact Hausdorff spaces. In this book, we take a somewhat more general point of view:

Definition 11 Let X and Y be sets, and let $C \subset 2^X$ and $D \subset 2^Y$ be algebras of subsets of X and Y, respectively. A scalar-valued set-function μ on $C \times D$ is an F_2-measure if for each $A \in C$, $\mu(A, \cdot)$ is a scalar measure on (Y, D), and for each $B \in D$, $\mu(\cdot, B)$ is a scalar measure on (X, C).

That bimeasures are F_2-measures is the two-dimensional extension of Theorem 2. (The utility of the more general definition is illustrated in Exercise 8.)

When highlighting the existence of 'true' bounded bilinear functionals, Morse and Transue all but ignored Littlewood's prior work. In their first paper on the subject, underscoring 'the difficult problem which Clarkson and Adams solve ... ', they stated [MorTr1, p. 155]: 'That [the Fréchet variation] can be finite while the classical total variation ... of Vitali is infinite has been shown by example by Clarkson and Adams [in [ClA]].' (In their 1933 paper [ClA], the authors did, in passing, attribute to Littlewood the first such example [ClA, p. 827], and then proceeded to give their own [ClA, pp. 837–41]. I prefer Littlewood's simpler example, which turned out to be more illuminating.) The more significant miss by Morse and Transue was a fundamental inequality that would play prominently in the bilinear theory – the same inequality that had been foreshadowed by Littlewood's earlier results.

3 More of the Bilinear

The inequality missed by Morse and Transue first appeared in Grothendieck's 1956 work [Gro2], a major milestone that was missed by most. The paper, pioneering new tensor-theoretic technology, was difficult to read and was hampered by limited circulation. (It was published in a journal carried by only a few university libraries.) The inequality itself, the highlight of Grothendieck's 1956 paper, was eventually unearthed a decade or so later. Recast and reformulated in a Banach space setting, this inequality became the focal point in a seminal 1968 paper by Lindenstrauss and Pelczynski [LiPe]. The impact of this 1968 work was decisive. Since then, the inequality, which Grothendieck himself billed as the 'théorème fondamental de la théorie metrique des produits tensoriels' has been reinterpreted and broadly applied in various contexts of analysis. It has indeed become recognized as a fundamental cornerstone.

Theorem 12 (the Grothendieck inequality). *If* $\hat\beta = (\hat\beta(m,n) : (m,n) \in \mathbb{N}^2)$ *is a real-valued array, and* $\{\mathbf{x}_n\}$ *and* $\{\mathbf{y}_n\}$ *are finite subsets in* B_{l^2}*, then*

$$\left|\sum_{n,m} \hat\beta(m,\ n)\langle \mathbf{x}_m, \mathbf{y}_n\rangle\right| \leq \kappa_G\ \|\hat\beta\|_{F_2}, \tag{3.1}$$

where B_{l^2} *is the closed unit ball in* l^2, $\langle\cdot,\cdot\rangle$ *denotes the usual inner product in* l^2*, and* $\kappa_G > 1$ *is a universal constant.*

Restated (via Lemma 8), the inequality in (3.1) has a certain aesthetic appeal:

$$\sup\Bigg\{\Bigg|\sum_{m\in S, n\in T}\hat{\beta}(m,n)\langle \mathbf{x}_m, \mathbf{y}_n\rangle\Bigg| : \mathbf{x}_m \in l^2, \mathbf{y}_n \in l^2,$$

$$\|\mathbf{x}_m\|_2 \le 1,\ \|\mathbf{y}_n\|_2 \le 1,\ \text{finite } S\subset \mathbb{N},\ T\subset\mathbb{N}\Bigg\}$$

$$\le \kappa_G \sup\Bigg\{\Bigg|\sum_{m\in S, n\in T}\hat{\beta}(m,n)x_m y_n\Bigg| : x_m \in \mathbb{R},$$

$$y_n\in\mathbb{R},\ |x_m|\le 1,\ |y_n|\le 1,\ \text{finite } S\subset\mathbb{N},\ T\subset\mathbb{N}\Bigg\}. \tag{3.2}$$

So stated, the inequality says that products of scalars on the right side of (3.2) can be replaced, up to a universal constant, by the dot product in a Hilbert space. In this light, a question arises whether one can replace the dot product on the left side of (3.1) with, say, the dual action between vectors in the unit balls of l^p and $l^q, 1/p + 1/q = 1$ and $p \in [1, 2)$. The answer is *no* (Exercise 6).

Grothendieck did not explicitly write what had led him to his 'théorème fondamental', but did remark [Gro2, p. 66] that Littlewood's mixed-norm inequality (Theorem 10) was an instance of it (Exercise 5). The actual motivation not withstanding, the historical connections between Grothendieck's inequality, Morse's and Transue's bimeasures, Littlewood's inequality(ies), and Fréchet's 1915 work are apparent in this important consequence of Theorem 12.

Theorem 13 (the Grothendieck factorization theorem). *Let X be a locally compact Hausdorff space. If β is a bounded bilinear functional on $\mathrm{C}_0(X)$ (a bimeasure on $X\times X$), then there exist probability measures ν_1 and ν_2 on the Borel field of X such that for all $f\in \mathrm{C}_0(X), g\in \mathrm{C}_0(X)$,*

$$|\beta(f,g)| \le \kappa_{\mathrm{G}}\|\beta\|\|f\|_{\mathrm{L}^2(\nu_1)}\|g\|_{\mathrm{L}^2(\nu_2)}, \tag{3.3}$$

where $\kappa_G > 0$ is a universal constant, and

$$\|\beta\| = \sup\{|\beta(f,g)| : (f,g)\in B_{\mathrm{C}_0(X)}\times B_{\mathrm{C}_0(X)}\}.$$

This 'factorization theorem', which can be viewed as a two-dimensional surrogate for the 'one-dimensional' Radon–Nikodym theorem, has a far-reaching impact. A case for it will be duly made in this book.

4 From Bilinear to Multilinear and Fraction-linear

Up to this point we have focused on the bilinear theory. As our story unfolds in chapters to come, we will consider questions about extending 'one-dimensional' and 'two-dimensional' notions to other dimensions: higher as well as fractional. Some answers will be predictable and obvious, but some will reveal surprises. In this final section of the prologue, we briefly sketch the backdrop and preview some of what lies ahead.

The multilinear Fréchet theorem in its simplest guise is a straightforward extension of Theorem 7:

Theorem 14 *An n-linear functional β on c_0 is bounded if and only if $\|\hat{\beta}\|_{F_n} < \infty$, where $\hat{\beta}(k_1, \ldots, k_n) = \beta(\mathbf{e}_{k_1}, \ldots, \mathbf{e}_{k_n})$ and*

$$\|\hat{\beta}\|_{F_n} = \sup\left\{\left\| \sum_{k_1 \in T_1, \ldots, k_n \in T_n} \hat{\beta}(k_1, \ldots, k_n) r_{k_1} \otimes \cdots \otimes r_{k_n} \right\|_\infty : \text{finite sets } T_1 \subset \mathbb{N}, \ldots, T_n \subset \mathbb{N}\right\}. \tag{4.1}$$

Moreover, the n-linear action of β on c_0 is given by

$$\beta(f_1, \ldots, f_n) = \sum_{k_1} \cdots \left(\sum_{k_n} \hat{\beta}(k_1, \ldots, k_n) f_n(k_n) \right) \cdots f_1(k_1), \quad (f_1, \ldots, f_n) \in \mathrm{c}_0 \times \cdots \times \mathrm{c}_0. \tag{4.2}$$

Though predictable, the analogous general measure-theoretic version requires a small effort. (The proof is by induction.)

The extension of Littlewood's 4/3-inequality to higher (integer) dimensions is not altogether obvious. (So far that I know, Littlewood himself never addressed the issue.) This extension, needed in a harmonic-analytic context, was stated and first proved by G. Johnson and G. Woodward in [JWo]:

Theorem 15

$$\|\hat{\beta}\|_p < \infty \text{ for all } n\text{-arrays } \hat{\beta} \text{ with } \|\hat{\beta}\|_{F_n} < \infty \quad \text{if and only if } p \geq \frac{2n}{n+1}.$$

'One half' of this theorem could be found also in [Da, p. 33]. For his purpose in [Da], Davie called on Littlewood's mixed-norm inequality

(Theorem 10), but did not need the 4/3-inequality. Nevertheless, he stated the latter, and remarked in passing without supplying proof that 'it [was] not hard to extend Littlewood's result' to obtain

$$\|\hat{\beta}\|_{2n/(n+1)} \leq 3^{\frac{n-1}{2}} n^{\frac{n+1}{2n}} \|\hat{\beta}\|_{F_n}. \tag{4.3}$$

(Davie did not state that (4.3) was optimal.)

Davie's paper is interesting in our context not only for its connection with Littlewood's inequalities, but also for a discussion therein of a seemingly unrelated, then-open question concerning multidimensional extensions of the von-Neumann inequality. This particular question was subsequently answered in the negative by N. Varopoulos, who, en route, demonstrated that there was no general *trilinear* Grothendieck-type inequality. The latter result concerning feasibility of Grothendieck-type inequalities in higher dimensions is a crucial part of our story here, indeed leading back to questions about extensions of Littlewood's 4/3-inequality. I will not dwell here or anywhere else in the book on the original problem concerning the von-Neumann inequality. But I shall state here the question, not only for its role as a catalyst, but also because an interesting related problem remains open. It is worth a small detour.

The von-Neumann inequality asserts that if T is a contraction on a Hilbert space and p is a complex polynomial in one variable, then

$$\|p(T)\| \leq \|p\|_\infty := \sup\{|p(z)| : |z| \leq 1\}, \tag{4.4}$$

where $\|\cdot\|$ above denotes the operator norm. The two-dimensional extension of (4.4) asserts that if T_1 and T_2 are commuting contractions on a Hilbert space, and p is a complex polynomial in two variables, then

$$\|p(T_1, T_2)\| \leq \|p\|_\infty := \sup\{|p(z_1, z_2)| : |z_1| \leq 1, |z_2| \leq 1\}. \tag{4.5}$$

(These inequalities can be found in [NF, Chapter 1].) The question whether

$$\|p(T_1, \ldots, T_n)\| \leq \|p\|_\infty,$$

where $n \geq 3$, $T_1, \ldots, T_n$ are commuting contractions on a Hilbert space, and p is a complex polynomial in n variables, was resolved in the negative in [V4]. But a question remains open: for integers $n \geq 3$, are there $K_n > 0$ such that if $T_1, \ldots, T_n$ are commuting contractions on a Hilbert space, and p is a complex polynomial in n variables, then

$$\|p(T_1, \ldots, T_n)\| \leq K_n \|p\|_\infty? \tag{4.6}$$

Let us return to the general $2n/n+1$-inequality in Theorem 15. The arguments used to prove Littlewood's inequality(ies) start from the observation that Rademacher functions are independent in the basic sense manifested by (1.5). The analogous observation in a Fourier-analysis setting is that the lacunary exponentials $\{e^{i3^m x} \colon m \in \mathbb{N}\}$ on $[0, 2\pi) := \mathbf{T}$ are independent in a like sense. Specifically, if $\Sigma_m \hat{\alpha}(m)\, e^{i3^m x}$ is the Fourier series of a continuous function on $\mathbf{T}$, then $\Sigma_m |\hat{\alpha}(m)| < \infty$ (cf. (1.5)). This phenomenon had been noted first by S. Sidon in 1926 [Si1], and later gave rise to a general concept whose systematic study was begun by Walter Rudin in his classic 1960 paper [RU1]:

Definition 16 $F \subset \mathbb{Z}$ is a Sidon set if

$$f \in \mathrm{C}_F(\mathbf{T}) \Rightarrow \hat{f} \in l^1(F), \tag{4.7}$$

where $\mathrm{C}_F(\mathbf{T}) := \{f \in \mathrm{C}(\mathbf{T}) : \hat{f}(m) = 0 \text{ for } m \notin F\}$.

Note that the counterpoint to Sidon's theorem (asserting that $\{3^k : k \in \mathbb{N}\}$ is a Sidon set) is that Placherel's theorem is otherwise optimal; that is,

$$\hat{f} \in l^p(\mathbb{Z}) \text{ for } \textit{all } f \in \mathrm{C}(\mathbf{T}) \Leftrightarrow p \geq 2. \tag{4.8}$$

These two 'extremal' properties – Sidon's theorem at one end, and (4.8) at the other – lead naturally to a question: for arbitrary $p \in (1, 2)$, are there $F \subset \mathbb{Z}$ such that

$$\hat{f} \in l^q(F) \text{ for } \textit{all } f \in \mathrm{C}_F(\mathbf{T}) \Leftrightarrow q \geq p? \tag{4.9}$$

To make matters concise, we define the *Sidon exponent* of $F \subset \mathbb{Z}$ by

$$\sigma_F = \inf\{p : \|\hat{f}\|_p < \infty \text{ for all } f \in \mathrm{C}_F(\mathbf{T})\}. \tag{4.10}$$

(Two situations could arise: either $\|\hat{f}\|_{\sigma_F} < \infty$ for all $f \in \mathrm{C}_F(\mathbf{T})$, or there exists $f \in \mathrm{C}_F(\mathbf{T})$ with $\|\hat{f}\|_{\sigma_F} = \infty$. Later in the book we will distinguish between these two scenarios.) Let $E = \{3^k : k \in \mathbb{N}\}$, and define for integers, $n \geq 1$

$$E_n = \{\pm 3^{k_1} \pm \cdots \pm 3^{k_n} : (k_1, \ldots, k_n) \in \mathbb{N}^n\}. \tag{4.11}$$

Transported to a context of Fourier analysis, Theorem 15 implies

$$\hat{f} \in l^q(E_n) \text{ for } \textit{all } f \in \mathrm{C}_{E_n}(\mathbf{T}) \Leftrightarrow q \geq \frac{2n}{n+1}. \tag{4.12}$$

In particular,

$$\sigma_{E_n} = 2 \Big/ \left(1 + \frac{1}{n}\right), \quad n \in \mathbb{N}, \tag{4.13}$$

which leads to the p-Sidon set problem (see (4.9)): for *arbitrary* $p \in (1,2)$, are there $F \subset \mathbb{Z}$ such that $\sigma_F = p$? The resolution of this problem – it so turned out – followed a resolution of a seemingly unrelated problem, that of extending the Grothendieck inequality to higher dimensions.

The Grothendieck inequality (Theorem 12) is a general assertion about bounded bilinear forms on a Hilbert space: in Theorem 12, replace l^2 by a Hilbert space H, and the inner product $\langle \cdot, \cdot \rangle$ in l^2 by a bounded bilinear form on H. A question arises: is there $K > 0$ such that for all bounded trilinear functionals β on c_0, all bounded trilinear forms A on a Hilbert space H, and all finite subsets $\{\mathbf{x}_n\} \subset B_H$, $\{\mathbf{y}_n\} \subset B_H$, and $\{\mathbf{z}_n\} \subset B_H$,

$$\left| \sum_{k,n,m} \hat{\beta}(m,n,k)\ A(\mathbf{x}_k, \mathbf{y}_m, \mathbf{z}_n) \right| \leq K \|\hat{\beta}\|_{F_3}? \tag{4.14}$$

(Here and throughout, B_X denotes the closed unit ball of a normed linear space X.) The question was answered in the negative by Varopoulos [V4], who demonstrated the following. For $H = l^2(\mathbb{N}^2)$, and $\varphi \in l^\infty(\mathbb{N}^3)$, define

$$A_\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{k,m,n} \varphi(k,m,n)\ \mathbf{x}(k,m)\ \mathbf{y}(m,n)\ \mathbf{z}(k,n),$$

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in l^2(\mathbb{N}^2) \times l^2(\mathbb{N}^2) \times l^2(\mathbb{N}^2), \tag{4.15}$$

which, by Cauchy–Schwarz, is a bounded trilinear form on H with norm $\|\varphi\|_\infty$. By use of probabilistic estimates, Varopoulos proved the existence of φ for which there was no $K > 0$ such that (4.14) would hold with $A = A_\varphi$ and all bounded trilinear functionals β on c_0. But a question remained: were there *any* $\varphi \in l^\infty(\mathbb{N}^3)$ for which A_φ would satisfy (4.14) for all bounded trilinear functionals β on c_0?

In 1976 I gave a new proof of the Grothendieck inequality [Bl3]. The proof, cast in a harmonic-analysis framework, was extendible to multidimensional settings, and led eventually to characterizations of *projectively bounded* forms [Bl4]. (Projectively bounded forms are those that satisfy Grothendieck-type inequalities, as in (4.14).) We illustrate this characterization in the case of the trilinear forms in (4.15). Choose and fix an arbitrary two-dimensional enumeration of $E = \{3^k : k \in \mathbb{N}\}$, say $E = \{m_{ij} : (i,j) \in \mathbb{N}^2\}$ (any enumeration will do), and consider

$$E^{\frac{3}{2}} := \{(m_{ij}, m_{jk}, m_{ik}) : (i,j,k) \in \mathbb{N}^3\}. \tag{4.16}$$

We then have

Theorem 17 *For $\varphi \in l^\infty(\mathbb{N}^3)$, the trilinear form A_φ is projectively bounded if and only if there exists a regular Borel measure μ on $\mathbf{T}^3$ such that*

$$\hat{\mu}(m_{ij}, m_{jk}, m_{ik}) = \varphi(i,j,k), \quad (i,j,k) \in \mathbb{N}^3. \tag{4.17}$$

Therefore, the question whether there exist φ such that A_φ is not projectively bounded becomes the question: is $E^{3/2}$ a Sidon set in $\mathbb{Z}^3$? The answer is *no.*

In the course of verifying that $E^{\frac{3}{2}}$ is not a Sidon set, certain combinatorial features of it come to light, suggesting that $E^{3/2}$ is a '3/2-fold' Cartesian product of E. Indeed, following this cue, we arrive at a 6/5-inequality [Bl5], which, in effect, is a '3/2-linear' extension of the Littlewood (bilinear) 4/3-inequality. For a scalar 3-array $\hat{\beta} = (\hat{\beta}(i,j,k) : (i,j,k) \in \mathbb{N}^3)$, define (the '3/2-linear' version of the Fréchet variation)

$$\|\hat{\beta}\|_{F_{3/2}} = \sup\left\{ \left\| \sum_{i\in S, j\in T, k\in U} \hat{\beta}(i,j,k)\ r_{ij} \otimes r_{jk} \otimes r_{ik} \right\|_\infty : \text{finite sets } S \subset \mathbb{N}, T \subset \mathbb{N}, U \subset \mathbb{N} \right\}. \tag{4.18}$$

(Rademacher systems in (4.18) are indexed by $\mathbb{N}^2$.) The 6/5-inequality is

Theorem 18

$$\|\hat{\beta}\|_p < \infty \textit{ for all 3-arrays } \hat{\beta} \textit{ with } \|\hat{\beta}\|_{F_{3/2}} < \infty$$
$$\textit{if and only if } p \geq 6/5.$$

Transporting this inequality to a setting of Fourier analysis, we let

$$E_{3/2} = \{\pm\, m_{ij} \pm m_{jk} \pm m_{ik} : (i,j,k) \in \mathbb{N}^3\}, \tag{4.19}$$

where $\{m_{ij} \colon (i,j) \in \mathbb{N}^2\}$ is an enumeration of $\{e^{2\pi i 3^k t} \colon k \in \mathbb{N}\}$, and obtain that

$$\sigma_{E_{3/2}} = \frac{6}{5} = 2\Big/\left(1 + 1\Big/\left(\frac{3}{2}\right)\right) \quad \text{(cf. (4.12))}. \tag{4.20}$$

The assertion in (4.20) is a precise link between the harmonic-analytic index $\sigma_{E_{3/2}}$ and the 'dimension' 3/2, a purely combinatorial index. This link naturally suggests a formula relating the harmonic-analytic index of a general 'fractional Cartesian product' to its underlying dimension, and thus the solution of the p-Sidon set problem. This (and much more) will be detailed in good time. The prologue is over. Let us begin.

Exercises

1. i. (*The Hilbert inequality*). Prove that if $(a_n) \in B_{l^2}$ and $(b_n) \in B_{l^2}$ are finitely supported sequences, then

$$\left| \sum_{m \neq n} a_n b_m/(m-n) \right| \leq K,$$

where K is a universal constant.

ii. Applying the Hilbert inequality, reproduce Littlewood's proof of the assertion (on p. 164 of [Li]) that there exist $\hat{\beta} = (\hat{\beta}(m,n) : (m,n) \in \mathbb{Z}^2)$ such that $\|\hat{\beta}\|_{F_2} < \infty$ but $\|\hat{\beta}\|_1 = \infty$.

iii. Compute the infimum of the ps such that $\|\hat{\beta}\|_p < \infty$, where $\hat{\beta}$ is the array obtained in ii.

2. Here are two other proofs, using probability theory, that there exist arrays $\hat{\beta} = (\hat{\beta}(m,n) : (m,n) \in \mathbb{N}^2)$ with $\|\hat{\beta}\|_{F_2} < \infty$ and $\|\hat{\beta}\|_1 = \infty$.

i. (a) Let $\{X_n : n \in \mathbb{N}\}$ be a system of statistically independent standard normal variables on a probability space $(X, \mathfrak{A}, \mathbb{P})$. Show that for every positive integer N, there exists a finite partition $\{A_m : m = 1, \ldots, 2^N\}$ of $(X, \mathfrak{A})$ such that if $\hat{\beta}_N(m,n) = \frac{1}{n}\mathbf{E}\mathbf{1}_{A_m} X_n$ for $n = 1, \ldots, N$ and $m = 1, \ldots, 2^N$, and $\hat{\beta}_N(m,n) = 0$ for all other $(n,m) \in \mathbb{N}^2$ ($\mathbf{E}$ denotes expectation, and $\mathbf{1}$ denotes an indicator function), then

$$\|\hat{\beta}_N\|_{F_2} \leq D \text{ and } \|\hat{\beta}_N\|_1 \geq D \log N,$$

where $D > 0$ is an absolute constant.

(b) Use (a) to produce $\hat{\beta} = (\hat{\beta}(m,n) : (m,n) \in \mathbb{N}^2)$ such that $\|\hat{\beta}\|_{F_2} < \infty$ but $\|\hat{\beta}\|_1 = \infty$ (cf. Exercise 4 iv below). What can be said about $\|\hat{\beta}\|_p$ for $p > 1$?

ii. (a) For each $N > 0$, define

$$\hat{\beta}_N(\omega, n) = r_n(\omega)/N^{\frac{1}{2}} 2^N, \quad \omega \in \{-1,1\}^N, \ n \in [N].$$

Prove that $\|\hat{\beta}_N\|_{F_2} \leq 1$. Compute $\|\hat{\beta}_N\|_p$ for $p \geq 1$.

(b) Use (a) to produce $\hat{\beta} = (\hat{\beta}(m,n)\colon\ (m,n) \in \mathbb{N}^2)$ such that $\|\hat{\beta}\|_{F_2} < \infty$, $\|\hat{\beta}\|_1 = \infty$, and $\|\hat{\beta}\|_p < \infty$ for all $p > 1$.

(Do you see similarities between the constructions in Parts i and ii? Do you see a similarity between the construction in Part ii and Exercise 4 below?)

3. Verify that if $\hat{\beta} = (\hat{\beta}(m,n) : (m,n) \in \mathbb{N}^2)$ is a scalar array then $\|\hat{\beta}\|_2 \leq \|\hat{\beta}\|_{F_2}$.
4. For $N \in \mathbb{N}$, let $\mathbb{Z}_N = [N]$ (a compact Abelian group with addition modulo N). Consider the characters

$$\chi_n(k) = \mathrm{e}^{2\pi \mathrm{i} kn/N}, \quad n \in \mathbb{Z}_N, \ k \in \mathbb{Z}_N,$$

and the Haar measure

$$\nu\{k\} = \frac{1}{N}, \qquad k \in \mathbb{Z}_N.$$

For $f \in l^\infty(\mathbb{Z}_N)$, define the transform of f by

$$\hat{f}(n) = \sum_{k \in \mathbb{Z}_N} f(k)\overline{\chi_n(k)}\ \nu(k).$$

i. (Orthogonality of characters) For $m \in \mathbb{Z}_N$ and $n \in \mathbb{Z}_N$, prove

$$\sum_{k \in \mathbb{Z}_N} \chi_m(k)\ \overline{\chi_n(k)}\ \nu(k) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

ii. (Inversion formula, Parseval's formula, Plancherel's theorem) Prove that for $f \in l^\infty(\mathbb{Z}_N)$,

$$f(n) = \sum_{k \in \mathbb{Z}_N} \hat{f}(k)\ \chi_k(n), \qquad n \in \mathbb{Z}_N.$$

Conclude that if $f \in l^\infty(\mathbb{Z}_N)$ and $g \in l^\infty(\mathbb{Z}_N)$, then

$$\sum_{k \in \mathbb{Z}_N} f(k)\ \overline{g(k)}\ \nu(k) = \sum_{k \in \mathbb{Z}_N} \hat{f}(k)\ \overline{\hat{g}(k)},$$

and that if $f \in \mathrm{L}^2(\mathbb{Z}_N, \nu)$, then

$$\|f\|_{\mathrm{L}^2(\mathbb{Z}_N,\nu)} = \|\hat{f}\|_{l^2(\mathbb{Z}_N)}.$$

iii. Prove that the 2-array $\left(\frac{\mathrm{e}^{2\pi\mathrm{i}(mn/N)}}{\sqrt{N}} : (m,n) \in \mathbb{Z}_N \times \mathbb{Z}_N\right)$ represents an isometry of $l^2(\mathbb{Z}_N)$. Define

$$\hat{\beta}(m,n) = \begin{cases} \frac{\mathrm{e}^{2\pi\mathrm{i}(mn/N)}}{N^{3/2}} & \text{if } (m,n) \in \mathbb{Z}_N \times \mathbb{Z}_N \\ 0 & \text{otherwise,} \end{cases}$$

and verify that $\|\hat{\beta}\|_{F_2} \leq 1$.

iv. Prove there exists a scalar array $\hat{\beta}$ with $\|\hat{\beta}\|_{F_2} < \infty$ and

$$\|\hat{\beta}\|_p = \infty \text{ for all } p < 4/3.$$

5. Prove that Littlewood's mixed norm inequality (Theorem 10) is an instance of the Grothendieck inequality (Theorem 12).
6. Let $\hat{\beta}$ be the scalar array defined in Exercise 4 iii. Let $q \in (2, \infty)$ and evaluate

$$\sup\left\{ \left| \sum_{m\in S, n\in T} \hat{\beta}(m,n)\langle \mathbf{e}_m, \mathbf{y}_n\rangle \right| : \right.$$
$$\left. \mathbf{y}_n \in B_{l^q},\ \text{finite sets } S \subset \mathbb{N}, T \subset \mathbb{N} \right\}.$$

   What does your computation say about an extension of Littlewood's mixed norm inequality (Theorem 10)? In particular, prove that the inner product in Grothendieck's inequality cannot be replaced by the dual action between vectors in the unit balls of l^p and l^q, $1/p + 1/q = 1$ and $p \in [1, 2)$.
7. Prove that β is a bounded n-linear functional on c_0 if and only if

$$\sum_{k_1,\ldots,k_n} \hat{\beta}(k_1,\ldots,k_n) e^{i3^{k_1}x_1} \cdots e^{i3^{k_n}x_n}$$

   represents a continuous function on $\mathbf{T}^n$.
8. This exercise, providing yet another example of a function with bounded Fréchet variation and infinite total variation, is a prelude to the 'probabilistic' portion of the book.

   A stochastic process $\mathrm{W} = \{\mathrm{W}(t) : t \in [0,\infty)\}$ defined on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ is a *Wiener process* if it satisfies these properties:

   (a) for $0 \le s < t < \infty, \mathrm{W}(t) - \mathrm{W}(s)$ is a normal r.v. with mean zero and variance $t - s$;

   (b) for $0 \le t_0 < t_1 \cdots < t_m < \infty,\ \mathrm{W}(t_k) - \mathrm{W}(t_{k-1}),\ k = 1,\ldots,m,$ are independent.

   Let $\mathfrak{J}$ denote the algebra generated by the intervals

$$\{(s,t] : 0 \le s < t \le 1\},$$

   and let μ_{W} be the set-function on $\mathfrak{A} \times \{(s,t] : 0 \le s < t \le 1\}$ defined by

$$\mu_{\mathrm{W}}(A,(s,t]) = \mathbf{E}\mathbf{1}_A(\mathrm{W}(t) - \mathrm{W}(s)), \quad A \in \mathfrak{A}, \quad 0 \le s < t \le 1.$$

i. Extend μ_{W} by additivity to $\mathfrak{A} \times \mathfrak{J}$.

ii. Prove that μ_{W} is an F_2-measure on $\mathfrak{A} \times \mathfrak{J}$ which is uniquely extendible to an F_2-measure on $\mathfrak{A} \times \mathfrak{B}$, where $\mathfrak{B}$ is the Borel field in [0,1].

iii. Prove that μ_{W} cannot be extended to a measure on the σ-algebra generated by $\mathfrak{A} \times \mathfrak{B}$.

Hints for Exercises in Chapter I

1. i. Here is an outline of a proof using elementary Fourier analysis. First, compute the Fourier coefficients of $h(x) = x$ on $\mathbf{T}$. Let $f(x) = \Sigma_n \hat{f}(n)\, \mathrm{e}^{\mathrm{i}nx}$ and $g(x) = \Sigma_n \hat{g}(n)\, \mathrm{e}^{\mathrm{i}nx}$ be trigonometric polynomials, and observe that

$$\left| \int_{\mathbf{T}} x f(x) g(x) \mathrm{d}x \right| \leq \pi \|f\|_{\mathrm{L}^2} \|g\|_{\mathrm{L}^2}$$
$$= \pi \left(\sum_n |\hat{f}(n)|^2 \right)^{\frac{1}{2}} \left(\sum_n |\hat{g}(n)|^2 \right)^{\frac{1}{2}}.$$

To prove the Hilbert inequality, use spectral analysis of fg, and apply Parseval's formula to the integral on the left side.

ii. Littlewood let $a_n = b_n = 1/\sqrt{|n|}(\log|n|)^\alpha$ for $n \in \mathbb{N}$, where $1/2 < \alpha < 1$, and then defined $\hat{\beta}(m,n) = a_n b_m/(m-n)$ for $n \neq m$.

2. For $N > 0$, consider $E_i = \{X_i > 0\}$, $i \in [N]$, and then for $s = (s_1, \ldots, s_N) \in \{-1,1\}^N$, let

$$A_s = E_1^{s_1} \cap E_2^{s_2} \ldots \cap E_k^{s_k},$$

where $E_i^{s_i} = E_i$ if $s_i = 1$, and $E_i^{s_i} = (E_i)^c$ if $s_i = -1$.

3. Cf. Plancherel's theorem.

7. This exercise involves basic notions that are covered at length in Chapter VII.

8. See Remark iv in Chapter VI § 2.