CAMBRIDGE STUDIES IN
ADVANCED MATHEMATICS 71

ANALYSIS IN INTEGER AND FRACTIONAL DIMENSIONS

Already published

1 W.M.L. Holcombe *Algebraic automata theory*
2 K. Peterson *Ergodic theory*
3 P.T. Johnstone *Stone spaces*
4 W.H. Schikhof *Ultrametric calculus*
5 J.-P. Kahane *Some randon series of functions, 2nd edition*
6 H. Cohn *Introduction to the construction of class fields*
7 J. Lambek & P.J. Scott *Introduction to higher-order categorical logic*
8 H. Matsumura *Commutative ring theory*
9 C.B. Thomas *Characteristic classes and the cohomology of finite groups*
10 M. Aschbacher *Finite group theory*
11 J.L. Alperin *Local representation theory*
12 P. Koosis *The logarithmic integral I*
13 A. Pietsch *Eigenvalues and s-numbers*
14 S.J. Patterson *An introduction to the theory of the Riemann zeta-function*
15 H.J. Baues *Algebraic homotopy*
16 V.S. Varadarajan *Introduction to harmonic analysis on semisimple Lie groups*
17 W. Dicks & M. Dunwoody *Groups acting on graphs*
18 L.J. Corwin & F.P. Greenleaf *Representations of nilpotent Lie Groups and their applications*
19 R. Fritsch & R. Piccinini *Cellular structures in topology*
20 H. Klingen *Introductory lectures on Siegal modular forms*
21 P. Koosis *The logarithmic integral II*
22 M.J. Collins *Representations and characters of finite groups*
24 H. Kunita *Stochastic flows and stochastic differential equations*
25 P. Wojtaszczyk *Banach spaces for analysts*
26 J.E. Gilbert & M.A.M. Murray *Clifford algebras and Dirac operators in harmonic analysis*
27 A. Frohlich & M.J. Taylor *Algebraic number theory*
28 K. Goebal & W.A. Kirk *Topics in metric fixed point theory*
29 J.F. Humphreys *Reflection groups and Coxeter groups*
30 D.J. Benson *Representations and cohomology I*
31 D.J. Benson *Representations and cohomology II*
32 C. Allday & V. Puppe *Cohomological methods in transformation groups*
33 C. Soulé et al *Lectures on Arakelov geometry*
34 A. Ambrosetti & G. Prodi *A primer of nonlinear analysis*
35 J. Palis & F. Takens *Hyperbolicity, stability and chaos at homoclinic bifurcations*
36 M. Auslander, I. Reiten & S.O. Smalø *Representation theory of Artin algebras*
37 Y. Meyer *Wavelets and operators I*
38 C. Weibel *An introduction to homological algebra*
39 W. Bruns & J. Herzog *Cohen-Macaulay rings*
40 V. Snaith *Explicit Brauer induction*
41 G. Laumon *Cohomology of Drinfield modular varieties I*
42 E.B. Davies *Spectral theory and differential operators*
43 J. Diestel, H. Jarchow & A. Tonge *Absolutely summing operators*
44 P. Mattila *Geometry of sets and measures in Euclidean spaces*
45 R. Pinsky *Positive harmonic functions and diffusion*
46 G. Tenenbaum *Introduction to analytic and probabilistic number theory*
47 C. Peskine *An algebraic introduction to complex projective geometry I*
48 Y. Meyer & R. Coifman *Wavelets and operators II*
49 R. Stanley *Enumerative combinatorics I*
50 I. Porteous *Clifford algebras and the classical groups*
51 M. Audin *Spinning tops*
52 V. Jurdjevic *Geometric control theory*
53 H. Voelklein *Groups as Galois groups*
54 J. Le Potier *Lectures on vector bundles*
55 D. Bump *Automorphic forms*
56 G. Laumon *Cohomology of Drinfield modular varieties II*
57 D.M. Clark & B.A. Davey *Natural dualities for the working algebraist*
59 P. Taylor *Practical foundations of mathematics*
60 M. Brodmann & R. Sharp *Local cohomology*
61 J.D. Dixon, M.P.F. Du Sautoy, A. Mann & D. Segal *Analytic pro-p groups, 2nd edition*
62 R. Stanley *Enumerative combinatorics II*
64 J. Jost & X. Li-Jost *Calculus of variations*
68 Ken-iti Sato *Lévy processes and infinitely divisible distributions*

Analysis in Integer and Fractional Dimensions

Ron Blei
University of Connecticut

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS

The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, VIC 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2001

Printed in the United Kingdom at the University Press, Cambridge

Typeface CMR 10/13 *System* LaTeX [HBA]

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

Blei, R. C. (Ron C.)
Analysis in integer and fractional dimensions/Ron Blei.
p. cm. – (Cambridge studies in advanced mathematics; 71)
Includes bibliographical references and index.
ISBN 0 521 65084 4
1. Harmonic analysis. 2. Functional analysis. 3. Probabilities.
4. Inequalities (Mathematics) I. Title. II. Series.

QA403 .B54 2001
515′.2433–dc21 00-064229

ISBN 0 521 65084 4 hardback

Dedication:

To the memory of my father, Nicholas Blei (1916–1968),
my mother, Isabel Guth Blei (1921–1975),
and my sister, Maya Blei (1952–1982).

Contents

Preface

What the book is about

In 1976 I gave a new proof to the Grothendieck (two-dimensional) inequality. The proof, pushed a little further, yielded extensions of the inequality to higher dimensions. These extensions, in turn, revealed 'Cartesian products in fractional dimensions', and led in a setting of harmonic analysis to the solution of the (so-called) p-Sidon set problem. The solution subsequently gave rise to an index of *combinatorial dimension*, a general measurement of interdependence with connections to harmonic, functional, and stochastic analysis. In 1993 I was ready to tell the story, and began teaching topics courses about this work. The notes for these courses eventually became this book.

Broadly put, the book is about 'dimensionality'. There are several interrelated themes, sub-themes, variations on themes. But at its very core, there is the notion that when we do mathematics – whatever mathematics we do – we start with independent building blocks, and build our constructs. Or, from an observer's viewpoint – not that of a builder – we *assume* existence of building blocks, and study structures we see. In either case, these are the questions: How are building blocks used, or put together? How complex are the constructs we build, or the structures we observe? How do we gauge, or detect, complexity? The answers involve notions of dimension.

The book is a mix of harmonic analysis, functional analysis, and probability theory. Part text and part research monograph, it is intended for students (no age restriction), whose backgrounds include at least one year of graduate analysis: measure theory, some probability theory, and some functional and Fourier analysis. Otherwise, I start discussions at the very beginning, and try to maintain a self-contained format.

Although the book is about specific brands of analysis, it should be accessible, and – I hope – interesting to mathematicians of other persuasions. I try to convey a sense of a 'big picture', with emphasis on historical links and contextual perspectives. And I try very hard to stay focused, not to be encyclopedic, to stick to the story.

The fourteen chapters are described below. Each except the first starts with 'mise en scène' (the setting of a stage), and ends with exercises. Some exercises are routine, filling in missing details, and some are not. There are some exercises (starred) that I do not know how to do. In fact, there are questions throughout the book, not only in the exercise sections, which I did not answer; some are open problems of long standing, and some arise naturally as the tale unfolds. We start at the beginning ('... a very good place to start ...'), and proceed along marked paths, with pauses at the appropriate stops. We go first through integer dimensions, and, en route, collect problems concerning the gaps between integer dimensions. These problems are solved in the last part of the book. Although there is a story here, and readers are encouraged to start at the beginning, the chapters are by and large modular. A savvy reader could select a starting point, and read confidently; all interconnections are clearly posted.

I A Prologue: Mostly Historical

A historical backdrop and flowchart: how it came about, and how it developed. There are very few proofs, and these few are very easy.

II Three Classical Inequalities

Three inequalities: Khintchin's, Littlewood's, and Orlicz's. These, which are equivalent in a precise sense, mark first steps.

III A Fourth Inequality

Grothendieck's fundamental inequality. Three proofs are given; all three are elementary, and all three involve an 'upgraded' Khintchin inequality.

IV Elementary Properties of the Fréchet Variation – an Introduction to Tensor Products

The Fréchet variation is a multi-dimensional extension of the l^1-norm and is at the heart of the matter. Basic properties are observed. The framework of tensor products is a convenient and natural setting for the 'multi-dimensional' mathematics done here.

V The Grothendieck Factorization Theorem

A two-dimensional statement, an equivalent of the Grothendieck inequality, with key applications in harmonic and stochastic analysis (later in the book). A multi-dimensional version is derived, but open questions persist about 'factorizability' in higher dimensions.

VI An Introduction to Multidimensional Measure Theory

A set-function on a Cartesian product of algebras is a Fréchet measure if it is countably additive separately in each coordinate. The theory of Fréchet measures generalizes notions in Chapter IV. Some multi-dimensional properties extend one-dimensional analogs, and some reveal surprises. The emphasis in this chapter is on the predictable properties.

VII An Introduction to Harmonic Analysis

A distinct introduction to a venerable area. Harmonic analysis in the setting $\{-1,1\}^{\mathbb{N}}$, viewed from the ground up, as it starts from independent Rademacher characters and evolves to the full Walsh system. The focus is on measurements of this evolution. In this chapter, measurements calibrate discrete scales of integer dimensions, and involve the Bonami inequalities and the Littlewood inequalities; measurements gauge interdependence and complexity. Questions concerning feasibility of 'continuous' scales are answered in later chapters.

VIII Multilinear Extensions of the Grothendieck Inequality (via $\Lambda(2)$-uniformizability)

Characterizations of Grothendieck-type inequalities in dimensions greater than two. Proofs are cast in a framework of harmonic analysis, and are based, as in Chapter III, on 'upgraded' Khintchin inequalities. Characterizations involve spectral sets that in a later chapter are viewed as Cartesian products in fractional dimensions.

IX Product Fréchet measures

Product *Fréchet* measures are multidimensional versions of product measures. They are as basic and important in the general multidimensional theory as are their analogs in classical one-dimensional frameworks. Feasibility of these products is inextricably tied to Grothendieck-type inequalities.

X Brownian Motion and the Wiener Process

In science at large, Brownian motion broadly refers to phenomena whose measurements appear to fluctuate randomly. The Wiener process, in effect a limit of simple random walks, provides a mathematical model 'in a first approximation' (Wiener) for such phenomena. Framed in a classical probabilistic setting, the Wiener process and subsequent chaos processes are viewed and analyzed from this book's perspective. Among the main themes are: (1) the identification of chaos processes with Fréchet measures; (2) measurements of evolving stochastic interdependence and complexity; (3) measurements of increasing levels of randomness in random walks.

XI Integrators

A continuation of themes in the previous chapter. A generic identification of Fréchet measures with stochastic processes; stochastic integration in a framework of multidimensional measure theory. The Grothendieck factorization theorem and inequality play prominently in the general stochastic setting.

XII A '3/2-dimensional' Cartesian Product

Analysis of the simplest example of a fractionally-dimensional Cartesian product. Dimension is a gauge of interdependence between coordinates.

XIII Fractional Cartesian Products and Combinatorial Dimension

Precise connections between *combinatorial dimension* and exponents of interdependence in frameworks of harmonic analysis and probability theory. Existence of sets with arbitrarily prescribed combinatorial dimensions (fractional Cartesian products, random sets).

XIV The Last Chapter: Leads and Loose Ends

Some applications and assessments of 'fractional-dimensional' analysis in multidimensional measure theory, harmonic analysis, and stochastic analysis. Open questions and future lines.

Conventions and Notations

Whenever possible, I use language of standard graduate courses in analysis and probability theory. Choice of scalars alternates between real and complex scalars, and is appropriately announced. Conventions and notations are introduced as we go along; every now and then, I review them for the reader.

Here are two examples of conventions that may not be standard, and appear frequently. If n is a positive integer, then $[n]$ denotes the set $\{1, \ldots, n\}$. *Independence* – a recurring theme in the book – appears under several guises, and I explicitly distinguish between these. For example, I refer to *statistical* independence (the mainstay notion in classical probability theory), and to *functional* independence (defined in the sequel). And there are other notions of independence.

Acknowledgements

The mathematics in the book benefited from numerous communications over the years, some in writing, some in conversation, and some by collaboration. I thank, in particular, Michael Benedicks, Bela Bollobás, Lennart Carleson, John Fournier, Evarist Giné, Dick Gosselin, Jean-Pierre Kahane, Tom Körner, Sten Kaijser, Jerry Neuwirth, Yuval Peres, Gilles Pisier, Jim Schmerl, Stu Sidney, Per Sjölin, Nick Varopoulos, and Moshe Zakai. (Some citations appear at various points in the text.)

Teaching topics courses was an integral part of the writing project – many thanks to my lively and loyal audiences for the active interest and the useful feedback. Special thanks to my Ph.D. students, who kept the enterprise going: Jay Caggiano, Fuchang Gao, Slaven Stricevic, and Nasser Towghi.

There were a few places in the book where, telling tales and waxing philosophical, I needed help. Warm thanks to my daughter Micaela, son David, and wife Judy for providing a true sounding board, for the good advice on style and tone, and for their love.

The completion of the book has been long overdue. My appreciation to Roger Astley, Miranda Fyfe, and the other good people at Cambridge University Press, for their unbounded patience, and for their excellent editorial work.