I

A Prologue: Mostly Historical

1 From the Linear to the Bilinear

At the start and at the very foundation, there is the *Riesz representation theorem.* In original form it is

Theorem 1 (F. Riesz, 1909). *Every bounded, real-valued linear functional α on $\mathrm{C}([a,b])$ can be represented by a real-valued function g of bounded variation on $[a,b]$, such that*

$$\alpha(f) = \int_a^b f \, \mathrm{d}g, \qquad f \in \mathrm{C}([a,b]), \tag{1.1}$$

where the integral in (1.1) is a Riemann–Stieltjes integral.

The measure-theoretic version, headlined also the *Riesz representation theorem*, effectively marks the beginning of functional analysis. In general form, it is

Theorem 2 *Let X be a locally compact Hausdorff space. Every bounded, real-valued linear functional on $\mathrm{C}_0(X)$ can be represented by a regular Borel measure ν on X, such that*

$$\alpha(f) = \int_X f \, \mathrm{d}\nu, \qquad f \in \mathrm{C}_0(X). \tag{1.2}$$

And in its most primal form, measure-theoretic (and non-trivial!) details aside, the theorem is simply

Theorem 3 *If α is a real-valued, bounded linear functional on* $\mathrm{c}_0(\mathbb{N}) = \mathrm{c}_0$*, then*

$$\|\hat{\alpha}\|_1 := \sum_n |\hat{\alpha}(n)| < \infty, \tag{1.3}$$

and

$$\alpha(f) = \sum_n \hat{\alpha}(n) f(n), \quad f \in \mathrm{c}_0,$$

where $\hat{\alpha}(n) = \alpha(\mathbf{e}_n)$ $(\mathbf{e}_n(n) = 1$, *and* $\mathbf{e}_n(j) = 0$ *for* $j \neq n)$.

The proof of Theorem 3 is merely an observation, which we state in terms of the Rademacher functions.

Definition 4 A Rademacher system indexed by a set E is the collection $\{r_x : x \in E\}$ of functions defined on $\{-1,1\}^E$, such that for $x \in E$

$$r_x(\omega) = \omega(x), \quad \omega \in \{-1,1\}^E. \tag{1.4}$$

To obtain the first line in (1.3), note that

$$\sup\left\{\left\|\sum_{n=1}^N \hat{\alpha}(n)\ r_n\right\|_\infty : N \in \mathbb{N}\right\} = \|\hat{\alpha}\|_1, \tag{1.5}$$

and to obtain the second, use the fact that finitely supported functions on $\mathbb{N}$ are norm-dense in $\mathrm{c}_0(\mathbb{N})$.

Soon after F. Riesz had established his characterization of bounded *linear* functionals, M. Fréchet succeeded in obtaining an analogous characterization in the *bilinear* case. (Fréchet announced the result in 1910, and published the details in 1915 [Fr]; Riesz's theorem had appeared in 1909 [Ri$_\mathrm{f}$1].) The novel feature in Fréchet's characterization was a two-dimensional extension of the *total variation* in the sense of Vitali. To wit, if f is a real-valued function on $[a,b] \times [a,b]$, then the *total variation* of f can be expressed as

$$\sup\left\{\left\|\sum_{n,m} \Delta^2 f(x_n, y_m)\ r_{nm}\right\|_\infty : a < \cdots < x_n < \cdots < b, \right.$$
$$\left. a < \cdots < y_m < \cdots < b\right\}, \tag{1.6}$$

where Δ^2 is the 'second difference',

$$\Delta^2 f(x_n, y_m) \\ = f(x_n, y_m) - f(x_{n-1}, y_m) + f(x_{n-1}, y_{m-1}) - f(x_n, y_{m-1}), \qquad (1.7)$$

and $\{r_{nm} : (n,m) \in \mathbb{N}^2\}$ is the Rademacher system indexed by $\mathbb{N}^2$. The two-dimensional extension of this one-dimensional measurement is given by:

Definition 5 The Fréchet variation of a real-valued function f on $[a,b] \times [a,b]$ is

$$\|f\|_{F_2} = \sup\left\{ \left\| \sum_{n,m} \Delta^2 f(x_n, y_m)\; r_n \otimes r_m \right\|_\infty : a < \cdots < x_n < \cdots < b, \right.$$
$$\left. < \cdots < y_m < \cdots < b \right\}. \qquad (1.8)$$

($r_n \otimes r_m$ is defined on $\{-1,1\}^{\mathbb{N}} \times \{-1,1\}^{\mathbb{N}}$ by

$$r_n \otimes r_m(\omega_1, \omega_2) = \omega_1(n)\omega_2(m),$$

and $\|\cdot\|_\infty$ is the supremum over $\{-1,1\}^{\mathbb{N}} \times \{-1,1\}^{\mathbb{N}}$.)

Based on (1.8), the bilinear analog of Riesz's theorem is

Theorem 6 (Fréchet, 1915). *A real-valued bilinear functional β on $\mathrm{C}([a,b])$ is bounded if and only if there is a real-valued function h on $[a,b] \times [a,b]$ with $\|h\|_{F_2} < \infty$, and*

$$\beta(f,g) = \int_a^b \int_a^b f \otimes g \;\mathrm{d}h, \quad f \in \mathrm{C}([a,b]),\; g \in \mathrm{C}([a,b]), \qquad (1.9)$$

where the right side of (1.9) *is an iterated Riemann–Stieltjes integral.*

The crux of Fréchet's proof was a construction of the integral in (1.9), a non-trivial task at the start of the twentieth century when integration theories had just begun developing.

Like Riesz's theorem, Fréchet's theorem can also be naturally recast in the setting of locally compact Hausdorff spaces; we shall come to this in good time. At this juncture we will prove only its primal version.

Theorem 7 *If β is a bounded bilinear functional on c_0, and $\beta(\mathbf{e}_m, \mathbf{e}_n) := \hat{\beta}(m,n)$, then*

$$\sup\left\{\left\|\sum_{m\in S, n\in T} \hat{\beta}(m,n)\ r_m \otimes r_n\right\|_\infty : \textit{finite sets } S \subset \mathbb{N}, T \subset \mathbb{N}\right\}$$

$$:= \|\hat{\beta}\|_{F_2} < \infty, \tag{1.10}$$

and

$$\begin{aligned} \beta(f,g) &= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\hat{\beta}(m,n)\ g(n)\right) f(m) \\ &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\hat{\beta}(m,n)\ f(m)\right) g(n), \\ & \qquad f \in \mathrm{c}_0, \quad g \in \mathrm{c}_0. \end{aligned} \tag{1.11}$$

Conversely, if $\hat{\beta}$ is a real-valued function on $\mathbb{N}\times\mathbb{N}$ such that $\|\hat{\beta}\|_{F_2} < \infty$, then (1.11) defines a bounded bilinear functional on c_0.

The key to Theorem 7 is

Lemma 8 *If $\hat{\beta} = (\hat{\beta}(m,n) : (m,n) \in \mathbb{N}^2)$ is a scalar array, then*

$$\begin{aligned} \|\hat{\beta}\|_{F_2} = \sup\Bigg\{\Bigg|\sum_{m\in S, n\in T}\hat{\beta}(m,n)\, x_m\, y_n\Bigg| &: x_m \in [-1,1], \\ & y_n \in [-1,1],\ \textit{finite sets } S \subset \mathbb{N}, T \subset \mathbb{N}\Bigg\}. \end{aligned} \tag{1.12}$$

Proof: The right side obviously bounds $\|\hat{\beta}\|_{F_2}$. To establish the reverse inequality, suppose S and T are finite subsets of $\mathbb{N}$, and $\omega \in \{-1,1\}^{\mathbb{N}}$. Then

$$\begin{aligned} \|\hat{\beta}\|_{F_2} &\geq \left\|\sum_{n\in T, m\in S}\hat{\beta}(m,n)\ r_m \otimes r_n\right\|_\infty, \\ &\geq \sum_{n\in T}\left|\sum_{m\in S}\hat{\beta}(m,n)\ r_m(\omega)\right|. \end{aligned} \tag{1.13}$$

If $y_n \in [-1,1]$ for $n \in T$, then the right side of (1.13) bounds

$$\left|\sum_{n\in T}\left(\sum_{m\in S}\hat{\beta}(m,n)\; r_m(\omega)\right) y_n\right| = \left|\sum_{m\in S}\left(\sum_{n\in T}\hat{\beta}(m,n)\; y_n\right) r_m(\omega)\right|. \tag{1.14}$$

By maximizing the right side of (1.14) over $\omega \in \{-1,1\}^{\mathbb{N}}$, we conclude that $\|\hat{\beta}\|_{F_2}$ bounds

$$\sum_{m\in S}\left|\sum_{n\in T}\hat{\beta}(m,n)\; y_n\right|. \tag{1.15}$$

If $x_m \in [-1,1]$ for $m \in S$, then (1.15) bounds

$$\begin{aligned}&\left|\sum_{m\in S}\left(\sum_{n\in T}\hat{\beta}(m,n)\; y_n\right) x_m\right|\\ &= \left|\sum_{m\in S, n\in T}\hat{\beta}(m,n)\, x_m\, y_n\right|, \end{aligned}\tag{1.16}$$

which implies that $\|\hat{\beta}\|_{F_2}$ bounds the right side of (1.12). □

Proof of Theorem 7: If β is a bilinear functional on c_0, with norm $\|\beta\| := \sup\{|\beta(f,g)| : f \in B_{\mathrm{c}_0},\ g \in B_{\mathrm{c}_0}\}$, then (because finitely supported functions are norm-dense in c_0)

$$\begin{aligned}\|\beta\| = \sup\Bigg\{&\left|\sum_{m\in S, n\in T}\hat{\beta}(m,n)\; x_m\; y_n\right| : x_m \in [-1,1],\\ &y_n \in [-1,1],\ \text{finite sets } S \subset \mathbb{N}, T \subset \mathbb{N}\Bigg\},\end{aligned}$$

and Lemma 8 implies (1.10).

Let $f \in \mathrm{c}_0$ and $g \in \mathrm{c}_0$. If $N \in \mathbb{N}$, then let $f_N = f\mathbf{1}_{[N]}$ and $g_N = g\mathbf{1}_{[N]}$. (Here and throughout, $[N] = \{1,\ldots,N\}$.) Because $f_N \to f$ and $g_N \to g$ as $N \to \infty$ (convergence in c_0), and β is continuous in each coordinate, we obtain $\beta(f_N, g) \to \beta(f,g)$ and $\beta(f, g_N) \to \beta(f,g)$ as $N \to \infty$, and then obtain (1.11) by noting that $\beta(f_N, g_N) = \Sigma_{m=1}^{N}\Sigma_{n=1}^{N}\hat{\beta}(m,n)g(n)f(m)$.

Conversely, if $\hat{\beta}$ is a scalar array on $\mathbb{N}\times\mathbb{N}$, and f and g are finitely supported real-valued functions on $\mathbb{N}$, then define

$$\beta(f,g) = \sum_m \sum_n \hat{\beta}(m,n)g(n)f(m). \tag{1.17}$$

By Lemma 8 and the assumption $\|\hat{\beta}\|_{F_2} < \infty$, β is a bounded bilinear functional on a dense subspace of c_0, and therefore determines a bounded bilinear functional on c_0. The first part of the theorem implies (1.10) and (1.11).

Theorem 7 was elementary, basic, and straightforward – view it as a warm-up. In passing, observe that whereas every bounded linear functional on c_0 obviously extends to a bounded linear functional on l^∞, the analogous fact in two dimensions, that every bounded bilinear functional on c_0 extends to a bounded bilinear functional l^∞ is also elementary, but not quite as easy to verify. This 'two-dimensional' fact, specifically that (1.11) extends to f and g in l^∞, will be verified in a later chapter.

2 A Bilinear Theory

Notably, Fréchet did not consider in his 1915 paper the question whether there exist functions with bounded variation in his sense, but with infinite total variation in the sense of Vitali. Whether *bilinear* functionals on $C([a, b])$ can be distinguished from *linear* functionals on $C([a, b]^2)$ is indeed a basic and important issue (Exercises 1, 2, 4, 8). So far as I can determine, Fréchet never considered or raised it (at least, not in print). Be that as it may, this question led directly to the next advance.

Littlewood began his classic 1930 paper [Lit4] thus: 'Professor P.J. Daniell recently asked me if I could find an example of a function of two variables, of bounded variation according to a certain definition of Fréchet, but not according to the usual definition.' Noting that the problem was equivalent to finding real-valued arrays

$$\hat{\beta} = (\hat{\beta}(m, n) : (m, n) \in \mathbb{N}^2)$$

with $\|\hat{\beta}\|_{F_2} < \infty$ and $\|\hat{\beta}\|_1 = \Sigma_{m,n}|\hat{\beta}(m, n)| = \infty$, Littlewood settled the problem by a quick use of the Hilbert inequality (Exercise 1). He then considered this question: whereas there are $\hat{\beta}$ with $\|\hat{\beta}\|_{F_2} < \infty$ and $\|\hat{\beta}\|_1 = \infty$, and (at the other end) $\|\hat{\beta}\|_{F_2} < \infty$ implies $\|\hat{\beta}\|_2 < \infty$ (Exercise 3), are there $p \in (1, 2)$ such that

$$\|\hat{\beta}\|_{F_2} < \infty \Rightarrow \|\hat{\beta}\|_p < \infty?$$

Littlewood gave this precise answer.

Theorem 9 (the 4/3 inequality, 1930).

$$\|\hat{\beta}\|_p < \infty \ \ \textit{for all} \ \ \hat{\beta} \textit{ with } \|\hat{\beta}\|_{F_2} < \infty \ \ \textit{if and only if} \ \ p \geq \frac{4}{3}.$$

To establish 'sufficiency', that $\|\hat{\beta}\|_{F_2} < \infty$ implies $\|\hat{\beta}\|_{4/3} < \infty$, Littlewood proved and used the following:

Theorem 10 (the mixed (l^1, l^2)-norm inequality, 1930). *For all real-valued arrays $\hat{\beta} = (\hat{\beta}(m,n) : (m,n) \in \mathbb{N}^2)$,*

$$\sum_m \left(\sum_n |\hat{\beta}(m,n)|^2 \right)^{\frac{1}{2}} \leq \kappa \|\hat{\beta}\|_{F_2}, \tag{2.1}$$

where $\kappa > 0$ is a universal constant.

This mixed-norm inequality, which was at the heart of Littlewood's argument, turned out to be a precursor (if not a catalyst) to a subsequent, more general inequality of Grothendieck. We shall come to Grothendieck's inequality in a little while.

To prove 'necessity', that there exists $\hat{\beta}$ with $\|\hat{\beta}\|_{F_2} < \infty$ and

$$\|\hat{\beta}\|_p = \infty \text{ for all } p < 4/3,$$

Littlewood used the *finite Fourier transform.* (You are asked to work this out in Exercise 4, which, like Exercise 1, illustrates first steps in harmonic analysis.)

Besides motivating the inequalities we have just seen, Fréchet's 1915 paper led also to studies of 'bilinear integration', first by Clarkson and Adams in the mid-1930s (e.g., [ClA]), and then by Morse and Transue in the late 1940s through the mid-1950s (e.g., [Mor]). For their part, firmly believing that the two-dimensional framework was interesting, challenging, *and* important, Morse and Transue launched extensive investigations of what they dubbed *bimeasures*: bounded bilinear functionals on $\mathrm{C}_0(X) \times \mathrm{C}_0(Y)$, where X and Y are locally compact Hausdorff spaces. In this book, we take a somewhat more general point of view:

Definition 11 Let X and Y be sets, and let $\mathcal{C} \subset 2^X$ and $\mathcal{D} \subset 2^Y$ be algebras of subsets of X and Y, respectively. A scalar-valued set-function μ on $\mathcal{C} \times \mathcal{D}$ is an F_2-measure if for each $A \in \mathcal{C}$, $\mu(A, \cdot)$ is a scalar measure on $(Y, \mathcal{D})$, and for each $B \in \mathcal{D}$, $\mu(\cdot, B)$ is a scalar measure on $(X, \mathcal{C})$.

That bimeasures are F_2-measures is the two-dimensional extension of Theorem 2. (The utility of the more general definition is illustrated in Exercise 8.)

When highlighting the existence of 'true' bounded bilinear functionals, Morse and Transue all but ignored Littlewood's prior work. In their first paper on the subject, underscoring 'the difficult problem which Clarkson and Adams solve . . . ', they stated [MorTr1, p. 155]: 'That [the Fréchet variation] can be finite while the classical total variation . . . of Vitali is infinite has been shown by example by Clarkson and Adams [in [ClA]].' (In their 1933 paper [ClA], the authors did, in passing, attribute to Littlewood the first such example [ClA, p. 827], and then proceeded to give their own [ClA, pp. 837–41]. I prefer Littlewood's simpler example, which turned out to be more illuminating.) The more significant miss by Morse and Transue was a fundamental inequality that would play prominently in the bilinear theory – the same inequality that had been foreshadowed by Littlewood's earlier results.

3 More of the Bilinear

The inequality missed by Morse and Transue first appeared in Grothendieck's 1956 work [Gro2], a major milestone that was missed by most. The paper, pioneering new tensor-theoretic technology, was difficult to read and was hampered by limited circulation. (It was published in a journal carried by only a few university libraries.) The inequality itself, the highlight of Grothendieck's 1956 paper, was eventually unearthed a decade or so later. Recast and reformulated in a Banach space setting, this inequality became the focal point in a seminal 1968 paper by Lindenstrauss and Pelczynski [LiPe]. The impact of this 1968 work was decisive. Since then, the inequality, which Grothendieck himself billed as the 'théorème fondamental de la théorie metrique des produits tensoriels' has been reinterpreted and broadly applied in various contexts of analysis. It has indeed become recognized as a fundamental cornerstone.

Theorem 12 (the Grothendieck inequality). *If $\hat{\beta} = (\hat{\beta}(m,n) : (m,n) \in \mathbb{N}^2)$ is a real-valued array, and $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ are finite subsets in B_{l^2}, then*

$$\left| \sum_{n,m} \hat{\beta}(m,\ n)\langle \mathbf{x}_m, \mathbf{y}_n \rangle \right| \leq \kappa_G \ \|\hat{\beta}\|_{F_2}, \tag{3.1}$$

where B_{l^2} is the closed unit ball in l^2, $\langle \cdot, \cdot \rangle$ denotes the usual inner product in l^2, and $\kappa_G > 1$ is a universal constant.

Restated (via Lemma 8), the inequality in (3.1) has a certain aesthetic appeal:

$$\sup\left\{\left|\sum_{m\in S,n\in T}\hat{\beta}(m,n)\langle\mathbf{x}_m,\mathbf{y}_n\rangle\right| : \mathbf{x}_m\in l^2, \mathbf{y}_n\in l^2,\right.$$

$$\left.\|\mathbf{x}_m\|_2\le 1,\ \|\mathbf{y}_n\|_2\le 1,\ \text{finite } S\subset\mathbb{N},\ T\subset\mathbb{N}\right\}$$

$$\le \kappa_G \sup\left\{\left|\sum_{m\in S,n\in T}\hat{\beta}(m,n)x_m y_n\right| : x_m\in\mathbb{R},\right.$$

$$\left. y_n\in\mathbb{R},\ |x_m|\le 1,\ |y_n|\le 1,\ \text{finite } S\subset\mathbb{N},\ T\subset\mathbb{N}\right\}. \tag{3.2}$$

So stated, the inequality says that products of scalars on the right side of (3.2) can be replaced, up to a universal constant, by the dot product in a Hilbert space. In this light, a question arises whether one can replace the dot product on the left side of (3.1) with, say, the dual action between vectors in the unit balls of l^p and $l^q, 1/p + 1/q = 1$ and $p \in [1, 2)$. The answer is *no* (Exercise 6).

Grothendieck did not explicitly write what had led him to his 'théorème fondamental', but did remark [Gro2, p. 66] that Littlewood's mixed-norm inequality (Theorem 10) was an instance of it (Exercise 5). The actual motivation not withstanding, the historical connections between Grothendieck's inequality, Morse's and Transue's bimeasures, Littlewood's inequality(ies), and Fréchet's 1915 work are apparent in this important consequence of Theorem 12.

Theorem 13 (the Grothendieck factorization theorem). *Let X be a locally compact Hausdorff space. If β is a bounded bilinear functional on $\mathrm{C}_0(X)$ (a bimeasure on $X \times X$), then there exist probability measures ν_1 and ν_2 on the Borel field of X such that for all $f \in \mathrm{C}_0(X), g \in \mathrm{C}_0(X)$,*

$$|\beta(f,g)| \le \kappa_{\mathrm{G}}\|\beta\|\|f\|_{\mathrm{L}^2(\nu_1)}\|g\|_{\mathrm{L}^2(\nu_2)}, \tag{3.3}$$

where $\kappa_G > 0$ is a universal constant, and

$$\|\beta\| = \sup\{|\beta(f,g)| : (f,g) \in B_{\mathrm{C}_0(X)} \times B_{\mathrm{C}_0(X)}\}.$$

This 'factorization theorem', which can be viewed as a two-dimensional surrogate for the 'one-dimensional' Radon–Nikodym theorem, has a far-reaching impact. A case for it will be duly made in this book.

4 From Bilinear to Multilinear and Fraction-linear

Up to this point we have focused on the bilinear theory. As our story unfolds in chapters to come, we will consider questions about extending 'one-dimensional' and 'two-dimensional' notions to other dimensions: higher as well as fractional. Some answers will be predictable and obvious, but some will reveal surprises. In this final section of the prologue, we briefly sketch the backdrop and preview some of what lies ahead.

The multilinear Fréchet theorem in its simplest guise is a straightforward extension of Theorem 7:

Theorem 14 *An n-linear functional β on c_0 is bounded if and only if $\|\hat{\beta}\|_{F_n} < \infty$, where $\hat{\beta}(k_1, \ldots, k_n) = \beta(\mathbf{e}_{k_1}, \ldots, \mathbf{e}_{k_n})$ and*

$$\begin{aligned} \|\hat{\beta}\|_{F_n} \;=\; \sup\Bigg\{ \Bigg\| \sum_{k_1 \in T_1, \ldots, k_n \in T_n} \hat{\beta}(k_1, \ldots, k_n) r_{k_1} \otimes \cdots \otimes r_{k_n} \Bigg\|_\infty : \\ \text{finite sets } T_1 \subset \mathbb{N}, \ldots, T_n \subset \mathbb{N} \Bigg\}. \qquad (4.1) \end{aligned}$$

Moreover, the n-linear action of β on c_0 is given by

$$\begin{aligned} \beta(f_1, \ldots, f_n) = \sum_{k_1} \cdots \left(\sum_{k_n} \hat{\beta}(k_1, \ldots, k_n) f_n(k_n) \right) \cdots f_1(k_1), \\ (f_1, \ldots, f_n) \in \mathrm{c}_0 \times \cdots \times \mathrm{c}_0. \qquad (4.2) \end{aligned}$$

Though predictable, the analogous general measure-theoretic version requires a small effort. (The proof is by induction.)

The extension of Littlewood's 4/3-inequality to higher (integer) dimensions is not altogether obvious. (So far that I know, Littlewood himself never addressed the issue.) This extension, needed in a harmonic-analytic context, was stated and first proved by G. Johnson and G. Woodward in [JWo]:

Theorem 15

$$\|\hat{\beta}\|_p < \infty \text{ for all } n\text{-arrays } \hat{\beta} \text{ with } \|\hat{\beta}\|_{F_n} < \infty$$
$$\text{if and only if } p \geq \frac{2n}{n+1}.$$

'One half' of this theorem could be found also in [Da, p. 33]. For his purpose in [Da], Davie called on Littlewood's mixed-norm inequality