

Theory of Solidification

The processes of freezing and melting were present at the beginnings of the Earth and continue to affect the natural and industrial worlds. The solidification of a liquid or the melting of a solid involves a complex interplay of many physical effects. This book systematically presents the field of continuum solidification theory based on instability phenomena. An understanding of the physics is developed by using examples of increasing complexity with the object of creating a deep physical insight applicable to more complex problems.

Applied mathematicians, engineers, physicists and materials scientists will all find this volume of interest.

Stephen H. Davis is McCormick Professor and Walter P. Murphy Professor of Applied Mathematics at Northwestern University.

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I dedicate this book to the wonderful women in my life,
my mother Eva
and
my wife Suellen

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Preface

Materials Science is an extremely broad field covering metals, semiconductors, ceramics, and polymers, just to mention a few. Its study is dominated by the *fabrication* of specimens and the *characterization* of their properties. A relatively small portion of the field is devoted to *phase transformation*, the dynamic process by which in the present context a liquid is frozen or a solid is melted.

This book is devoted to the study of liquid (melt)-solid transformations of atomically rough materials: metals or semiconductors, including model organics like plastic crystals. The emphasis is on the use of instability behavior as a means of understanding those processes that ultimately determine the microstructure of a crystalline solid. The fundamental building block of this study is the Mullins–Sekerka instability of a front, which gives conditions for the growth of infinitesimal disturbances of a solid–liquid front. This is generalized in many ways: into the nonlinear regime, including thermodynamic disequilibrium, anisotropic material properties, and effects of convection in the liquid. Cellular, eutectic, and dendritic behaviors are discussed. The emphasis is on dynamic phenomena rather than equilibria. In a sense then, it concerns “physiology” rather than “anatomy.”

The aim of this book is to present in a systematic way the field of continuum solidification theory. This begins with the primitive field equations for diffusion and the derivation of appropriate jump conditions on the interface between the solid and liquid. It then uses such models to explore morphological instabilities in the linearized range and gives physical explanations for the phenomena uncovered. To this point the discussion is elementary in terms of mathematical sophistication. It then enters into the nonlinear theories of morphological change with the use of bifurcation theory for wave number and pattern selection, long-wave theories in the strongly nonlinear range, and numerical simulation. The reader is assumed to be reasonably sophisticated in the mathematical methods,

that is, stability theory and its nonlinear extensions and some asymptotic and perturbation theory, but having little background in materials science. Thus, the book is deliberately nonuniform in its “degree of difficulty.” Those with limited mathematical background can skip the nonlinear theories and read about the physical phenomena and the linearized theories in the various chapters. The text should take the reader from the elements of the physics to the latest developments of the theory. It would be hoped that applied mathematicians, engineers, and physicists would profit from the material presented as would theoretically inclined materials scientists who could see how mathematics can generate understanding of relevant physical phenomena. An understanding of the physics is developed by using examples of increasing complexity with the objective of creating a deep physical insight applicable to more complex problems.

My interest in solidification was first stimulated by Jon Dantzig in his Ph.D. thesis of 1977 and permanently triggered by Ulrich Müller in our 1984 work on Bénard convection coupled to a freezing front. When learning a new subject as an “adult,” one leans heavily on the expertise of senior colleagues for their wisdom. I thus wish to publicly thank Sam Coriell, Jon Dantzig, Paul Fife, Marty Glicksman, Wilfried Kurz, Jeff McFadden, Uli Müller, Bob Sekerka, Peter Voorhees, and Grae Worster for their contributions to my education.

I have always learned more from my graduate students, post-doctoral fellows, and visiting scientists than they have from me. I wish to thank them for their willingness to try something new. They are V. S. Ajaev, K. Brattkus, R. J. Braun, L. Bühler, D. J. Canright, Y.-J. Chen, J. A. Dantzig, A. A. Golovin, H.-P. Grimm, D. A. Huntley, P.-Q. Luo, G. B. McFadden, G. J. Merchant, P. Metzener, U. Müller, D. S. Riley, T. P. Schulze, B. J. Spencer, A. Umantsev, G. W. Young, and J.-J. Xu.

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Finally, I would like to thank my secretary, Judy Piehl, not only for her impeccable typing, but for her sense of joy in her work. Her presence in the department makes it possible for all of us to do better what we do.