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The *q*-Schur Algebra

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I dislike arguments of any kind.
They are always vulgar, and often convincing.

Oscar Wilde, *The Importance of Being Earnest*

Contents

Preface

These notes are concerned with the representation theories of the quantum general linear groups, the q-Schur algebras and the Hecke algebras of type A, and, most importantly, the relationships between these theories. Roughly speaking, we are presenting a q-analogue of the monograph by J. A. Green, [**51**], which treats the representation theory of general linear groups, Schur algebras and symmetric groups. The theory developed here is a generalization of the classical case (which one recovers by putting $q = 1$). But the main difference between [**51**] and this text is that whereas Green's approach is combinatorial ours is (for the most part) homological, informed (or made possible even) by Kempf's vanishing theorem (for quantum GL_n). This approach is a continuation of one developed in our earlier papers on Schur algebras and related algebras, [**31**], [**32**], [**33**], [**34**], [**35**].

The version of quantum GL_n that we shall use is the one introduced by R. Dipper and the author. The q-Schur algebras were introduced by Dipper and James, [**21**], as endomorphism algebras of certain modules over Hecke algebras. Mostly, we work over an arbitrary field k and q is an element of k, which is usually required to be non-zero. For a positive integer n and a non-negative integer r, we have the Schur algebra $S_q(n,r)$. In the approach taken here (modelled on the treatment of ordinary Schur algebras by Green, [**51**]) $S_q(n,r)$ is constructed as the dual algebra of a certain coalgebra. More precisely, the coordinate algebra $k[G_q(n)]$ of the quantum general linear group $G_q(n)$ of degree n is generated by "coefficient functions" c_{ij}, $1 \leq i,j \leq n$, and the inverse of the quantum determinant. Thus the (in general non-commutative) algebra $A_q(n)$ generated by the c_{ij} is a subbialgebra of $k[G_q(n)]$, and $A_q(n)$ has an algebra grading and coalgebra decomposition $A_q(n) = \bigoplus_{r=0}^{\infty} A_q(n,r)$ in which each c_{ij} has degree 1. The coalgebra $A_q(n,r)$ has dimension $\binom{n^2+r-1}{r}$ and its dual algebra is $S_q(n,r)$. A module for $S_q(n,r)$ is naturally an $A_q(n,r)$-comodule and hence a $k[G_q(n)]$-comodule, i.e. a module for the quantum group $G_q(n)$. Thus the category of $S_q(n,r)$-modules is naturally embedded as a full subcategory of the category of $G_q(n)$-modules, namely the category of $G_q(n)$-modules which are polynomial of degree r.

Suppose now that $r \leq n$. Then there is a distinguished idempotent $e \in S_q(n,r)$ such that $eS_q(n,r)e$ is the Hecke algebra $\mathrm{Hec}(r)$ defined by the symmetric group $\mathrm{Sym}(r)$ of degree r (regarded as a Coxeter group). Thus one has, as in the classical case $q = 1$ (see Green, [**51**, Chapter 6]), the Schur functor from the category of $S_q(n,r)$-modules to the category of $\mathrm{Hec}(r)$-modules, taking an $S_q(n,r)$-module V to the subspace eV, viewed as a module for $\mathrm{Hec}(r) = eS_q(n,r)e$. Our philosophy here is to proceed uniformly in this direction: that is, to first prove results about our quantum version of GL_n, then to use this knowledge to deduce results about the q-Schur algebras and finally, by a further "descent", to obtain results on the Hecke algebra. Our purpose then is twofold: not only to present new results but also to rederive known results by these descents. Perhaps an extreme example of the latter, given in Section 4.3, is the determination of the labelling of irreducible modules for Hecke algebras, first obtained by Dipper and James, [**20**], by decomposing $E^{\otimes r}$, the rth tensor power of the natural module for quantum GL_n, viewed as a tilting module. An exception to this philosophy is Section 2.2, where we use the representation theory of the Hecke algebra at $q = 0$ to describe explicitly the characters of the irreducible modules for the 0-Schur algebras.

These notes started life as part of a manuscript which dealt with the standard homological properties of $G_q(n)$ as well as some other topics. We published the homological parts separately, [**36**], and prepared as a companion paper the other topics (as detailed in the last paragraph of the introduction of [**36**]). However, as time went by, we kept adding to this manuscript, and a desire to keep the material together and the prospect of an opportunity to present the material from the uniform point of view described above led to the existence of the notes in their present form. The original intention of publication as a research article is responsible for the terse journal style of the main body of the text (and the fact that these notes are referred to in various places as "On Schur algebras and related algebras VI: The q-Schur algebra"). We have tried to compensate for this, and to make the notes reasonably self contained, by adding a long expository introductory chapter, which starts with the representation theory of algebraic groups and makes a gradual transition to the representation theory of quantum GL_n, and also by adding an appendix on quasihereditary algebras.

We defer a more detailed description of the contents of the notes until the end of the introductory chapter, so as to avail ourselves of the notation and definitions given there.

I am grateful to the School of Mathematical Sciences (especially to the Algebraic Lie Theory seminar) of Queen Mary and Westfield College, University of London, for the opportunity to present various parts of these notes at various times and also to the Institute for Experimental Mathematics, Essen, for the opportunity to lecture there (on the results given in Section 4.1 and Section 4.2(14), on the Ringel dual of the q-Schur algebras) in April 1994.

I am grateful to Anton Cox for his help in detecting numerous minor errors in earlier versions of these notes.