0. Introduction

0.1 In this chapter we endeavour to take the reader slowly from the familiar world of rational representations of algebraic groups to that of rational representations of general linear quantum groups, q-Schur algebras and Hecke algebras of type A. We do this partly to stress the close analogies between the theories, partly to establish some notation and list in a convenient form some results which are to be used in the sequel, and partly so that we will be able to describe in outline, towards the end of this chapter, the contents of the main part of these notes. Appropriate references for each section are given at the end of the chapter.

0.2 We fix an algebraically closed field K and a subfield k. We begin by recalling the basics of the theory of affine varieties and linear algebraic groups over K. Suppose given a set V and an algebra R of K-valued functions on V. For each point $x \in V$ we have the evaluation map $\varepsilon_x : R \to K$, defined by $\varepsilon_x(f) = f(x)$, for $f \in R$. The pair (V, R) is called an *affine variety* (over K) if the algebra R is finitely generated over K and the map $x \mapsto \varepsilon_x$, from V to the set $\mathrm{Hom}_{K-\mathrm{alg}}(R, K)$, of K-algebra homomorphisms from R to K, is bijective. An affine variety (V, R) is usually abbreviated to V. The algebra R, usually written $K[V]$, is called the coordinate algebra of V, or algebra of regular functions on V.

If (V, R) and (W, S) are affine varieties, a map $\phi : V \to W$ is a *morphism* of affine varieties if we have $g \circ \phi \in R$ for every $g \in S$. Such a map determines a K-algebra homomorphism $\phi^* : S \to R$, given by $\phi^*(g) = g \circ \phi$, and conversely, one checks that each K-algebra homomorphism $\theta : S \to R$ may be written $\theta = \phi^*$ for a unique morphism $\phi : V \to W$. For a subset Z of V we define $I_Z = \{f \in R \mid f(x) = 0 \text{ for all } x \in Z\}$. A subset Z is *closed* if there exist regular functions $f_1, \ldots, f_r$ on V such that $Z = \{x \in V \mid f_1(x) = \cdots = f_r(x) = 0\}$. The closed sets of V form the closed sets of the *Zariski topology* of V. If Z is closed in V then Z is naturally an affine variety with coordinate algebra $K[V]/I_Z$. (That is, the regular functions on Z are just the restriction to Z of regular functions on V.)

We write $\mathbb{A}^n$ for $K \times \cdots \times K$ (n times). For $1 \leq i \leq n$ we have the coordinate function $X_i : \mathbb{A}^n \to K$, defined by $X_i(x) = x_i$, for $x = (x_1, \ldots, x_n) \in \mathbb{A}^n$. The functions $X_1, \ldots, X_n$ are algebraically independent over K and one quickly verifies that $(\mathbb{A}^n, K[X_1, \ldots, X_n])$ is an affine algebraic variety. Moreover if V is an algebraic variety, with coordinate algebra A generated by $a_1, \ldots, a_r$, then the algebra map $\theta : K[X_1, \ldots, X_r] \to A$ given by $\theta(X_i) = a_i$, $1 \leq i \leq r$, corresponds to a morphism $\phi : V \to \mathbb{A}^r$ which identifies V with a closed subset of $\mathbb{A}^r$. (The image of ϕ is the set of $x \in \mathbb{A}^r$ such that $f(x) = 0$ for all f in the kernel of $\theta : K[X_1, \ldots, X_r] \to A$.) In this way we recover the usual description of affine K-varieties as closed subsets of affine r-space.

Given a finitely generated commutative K-algebra R without nilpotent elements (i.e. $f^m = 0$ implies $f = 0$, for $f \in R$ and $m > 0$) we may construct an affine variety with R as its coordinate algebra. For V we take $\mathrm{Hom}_{K-\mathrm{alg}}(R, K)$. For each $f \in R$ we have the function $\tilde{f} : V \to K$ defined by $\tilde{f}(x) = x(f)$, $x \in V$. We define $\tilde{R}$ to be the algebra of functions on V consisting of all $\tilde{f}$, $f \in R$. If $0 \neq f \in R$ we may choose a maximal ideal M of R not containing f. By the Nullstellensatz, inclusion $K \to R$ induces an isomorphism $K \to R/M$. Thus we have some K-algebra homomorphism $x : R \to K$ with kernel M and $x(f) \neq 0$. Hence the natural map $R \to \tilde{R}$ is injective and therefore an isomorphism. Identifying R with $\tilde{R}$ via this map we have that (V, R) is an affine variety. Thus the category of affine K-varieties is equivalent to the category of reduced (i.e. without nilpotent elements) finitely generated commutative K-algebras.

Let (V, R) and (W, S) be affine varieties. For $f \in R$, $g \in S$ we have the function $h_{f,g} : V \times W \to K$ defined by $h_{f,g}(x, y) = f(x)g(y)$, for $x \in V$, $y \in Y$. Let T be the algebra of K-valued functions on $V \times W$ generated by all such functions. The natural map $R \otimes_K S \to T$ (taking $f \otimes g$ to $h_{f,g}$) is an isomorphism and we thereby identify $R \otimes_K S$ with T. One checks that $(V \times W, T)$ is an affine K-variety and indeed this is the product of (V, R) and (W, S) in the category of affine varieties. (Alternatively, we could define $V \times W$ to be the affine variety whose coordinate algebra is $R \otimes_K S$, in view of the paragraph above.) Note that we have $\mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{m+n}$.

0.3 By a *linear algebraic group* over K we mean a group G which is also an affine K-variety in such a way that the structure maps $m : G \times G \to G$ (multiplication) and $i : G \to G$ (inversion) are morphisms of varieties. Consider the general linear group $\mathrm{GL}_n(K)$ of invertible $n \times n$ matrices. We define $c_{ij} : \mathrm{GL}_n(K) \to K$ to be the (i, j) coordinate function, $1 \leq i, j \leq n$. We define $d : \mathrm{GL}_n(K) \to K$ to be the determinant function. The coordinate functions c_{ij}, $1 \leq i, j \leq n$, are algebraically independent over K. The coordinate algebra $K[\mathrm{GL}_n(K)]$ is, by definition, the algebra of K-valued functions on $\mathrm{GL}_n(K)$ generated by all coordinate functions c_{ij} together with d^{-1} (the function taking $x \in \mathrm{GL}_n(K)$ to the reciprocal of the determinant of x). Then $\mathrm{GL}_n(K)$ is a linear algebraic group (with coordinate algebra $K[\mathrm{GL}_n(K)]$).

0.4 Let G be a linear algebraic group. We say that a matrix representation $\rho : G \to \mathrm{GL}_n(K)$ is *rational* if ρ is a homomorphism of linear algebraic groups (i.e. a group homomorphism and a morphism of varieties). We say that a finite dimensional KG-module V is *rational* if it affords a rational matrix representation with respect to some (and hence every) basis. Let us be quite explicit. Choose a K-basis $v_1, \ldots, v_n$ of V and define coefficient functions $f_{ij} : G \to K$ by the equations

$$xv_i = \sum_{j=1}^{n} f_{ji}(x)v_j \qquad (x \in G, 1 \leq i \leq n).$$

The *coefficient space* of V is defined to be the space of K-valued functions on G spanned by the coefficient functions f_{ij}, $1 \leq i, j \leq n$. This space is independent of the choice of basis and we denote it by $\mathrm{cf}(V)$. Then the condition for V to be rational is that $\mathrm{cf}(V)$ should consist of regular functions, i.e. $\mathrm{cf}(V) \leq K[G]$. Note that if V, W are finite dimensional KG-modules then we have $\mathrm{cf}(V \otimes_K W) = \mathrm{cf}(V).\mathrm{cf}(W)$, the K-span of all functions fg with $f \in \mathrm{cf}(V)$, $g \in \mathrm{cf}(W)$. In particular $V \otimes_K W$ is rational if V and W are rational. For a K-valued function f on G we write $\bar{f}$ for the K-valued function on G defined by the formula $\bar{f}(x) = f(x^{-1})$, $x \in G$. Note that $\bar{f} = i^*(f) \in K[G]$ if $f \in K[G]$. For a finite dimensional (left) KG-module V with coefficient space C we have that the coefficient space of the dual left module V^* is $\bar{C} = \{\bar{f} \mid f \in C\}$. It follows that if V is rational then so is V^*.

We say that a KG-module V of arbitrary dimension is rational if it is the union of finite dimensional rational submodules. If V is rational then so is every submodule and quotient module. A module is rational if it is generated by rational submodules.

0.5 Let G be a linear algebraic group over K. Then $K[G]$ is naturally a left KG-module for the left regular action, which we now describe. For $x \in G$, $f \in K[G]$ the function xf is given by the formula $(xf)(y) = f(yx)$, for $y \in G$. If $m^*(f) = \sum_{i=1}^n f_i \otimes f_i'$ then we have $xf = \sum_{i=1}^n f_i'(x) f_i$. In particular $xf \in K[G]$ and so $K[G]$ is naturally a left KG-module. We choose a basis $\{v_i \mid i \in I\}$ of $K[G]$. Then, for $i \in I$, we have $m^*(v_i) = \sum_{j \in I} v_j \otimes f_{ji}$ for elements f_{ij} of $K[G]$. We fix an $i \in I$. For $x \in G$ we have $xv_i = \sum_{j \in I} f_{ji}(x) v_j$. Since only finitely many of the f_{ji} are non-zero (for fixed i), for all $x \in G$ the element xv_i lies in the finite dimensional space spanned by the v_j for which f_{ji} is non-zero. Hence the KG-module V_i, say, generated by v_i, is finite dimensional. Let $W = V_i$ and let $w_1, \ldots, w_l$ be a basis of W. We have $m^*(w_r) = \sum_{s=1}^l w_s \otimes g_{sr}$, for elements $g_{rs} \in K[G]$ with $1 \leq r, s \leq l$. Hence $xw_r = \sum_{s=1}^l g_{sr}(x) w_s$ and the coefficient space of W is the K-span of the g_{rs}, in particular V_i is a rational module. But $K[G] = \sum_{i \in I} V_i$ and hence $K[G]$ is a rational G-module with respect to the left regular action.

0.6 An importance consequence of 0.5 is that every linear algebraic group is isomorphic to a closed subgroup of $\mathrm{GL}_n(K)$, for suitable n. Let G be a linear algebraic group. Let $a_1, \ldots, a_m$ be a set of algebra generators of $K[G]$ and let W be a finite dimensional subspace of $K[G]$ which contains these generators and is a submodule for the left regular G-module action. Let $w_1, \ldots, w_n$ be a basis of W and let f_{ij} be the corresponding coefficient functions. Thus we have $xw_i = \sum_{j=1}^n f_{ji}(x)w_j$, for $1 \le i \le n$. We define $\rho : G \to \mathrm{GL}_n(K)$ by $\rho(x) = (f_{ij}(x))$, $x \in G$. Then ρ is a group homomorphism. We claim that ρ is a morphism of varieties, and hence a morphism of algebraic groups. We have $c_{ij} \circ \rho = f_{ij}$, and hence $c \circ \rho \in K[G]$ for all c in the subalgebra of $K[\mathrm{GL}_n(K)]$ generated by all c_{ij}. Moreover, since $K[\mathrm{GL}_n(K)]$ is generated by all c_{ij} together with d^{-1}, it suffices to show that $d^{-1} \circ \rho \in K[G]$. So let $f = d \circ \rho$ and $g = d^{-1} \circ \rho$. Then we have $fg = 1$ and hence f is a regular function on G which is everywhere non-zero. However, it is a general fact that if h is a regular function on an affine variety V which is everywhere non-zero then $h^{-1} = 1/h$ is regular. (If not then the ideal of $K[V]$ generated by h is a proper ideal and hence contained in a maximal ideal M, say. By the Nullstellensatz, once more, M has codimension 1 in $K[V]$ and there is a K-algebra homomorphism $\theta : K[V] \to K$ which vanishes on M. But we have $\theta = \varepsilon_x$ for some $x \in V$ and $h(x) = \varepsilon_x(h) = \theta(h) = 0$, a contradiction.) Hence $g = d^{-1} \circ \rho \in K[G]$. Thus $\rho : G \to \mathrm{GL}_n(K)$ is indeed a morphism of algebraic groups. The image of ρ^* contains the generators $a_1, \ldots, a_m$ of $K[G]$. Thus $\rho^* : K[\mathrm{GL}_n(K)] \to K[G]$ is surjective and it follows that the image of ρ is closed in $\mathrm{GL}_n(K)$ and that ρ induces an isomorphism of algebraic groups from G onto the image of ρ.

0.7 Let k be an arbitrary field. We take tensor products over k. A *coalgebra* over k (or k-coalgebra) is a triple (C, δ, ε) consisting of a k-vector space C and linear maps $\delta : C \to C \otimes C$ (comultiplication) and $\varepsilon : C \to k$ (the counit or augmentation map) satisfying the equations

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta : C \to C \otimes C \otimes C$$

and

$$(\varepsilon \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \varepsilon) \circ \delta = \mathrm{id} : C \to C$$

where id denotes the identity map on C. The first equation is called the coassociativity condition and the second is called the counit condition.

A *bialgebra* over k (or k-bialgebra) is a coalgebra (A, δ, ε) such that A is a k-algebra and the structure maps $\delta : A \to A \otimes A$ and $\varepsilon : A \to k$ are algebra homomorphisms. A bialgebra (A, δ, ε) is called a *Hopf algebra* if there exists a linear map $\sigma : A \to A$ such that

$$\mu(\sigma \otimes \mathrm{id})\delta = \mu(1 \otimes \sigma)\delta = \tilde{\varepsilon}$$

where $\mu : A \otimes A \to A$ is the multiplication map and $\tilde{\varepsilon} : A \to A$ is defined by $\tilde{\varepsilon}(a) = \varepsilon(a)1$, for $a \in A$. The above equation is called the antipode condition.

If such a linear map $\sigma : A \to A$ exists, it is uniquely determined by the antipode condition and called the *antipode* of (A, δ, ϵ). A bialgebra (or Hopf algebra) is said to be commutative if the algebra A is commutative. In general the antipode of a Hopf algebra (A, δ, ε) is an algebra anti-homomorphism and hence is an algebra homomorphism if A is commutative. If (A, δ, ε), $(A', \delta', \varepsilon')$ are coalgebras, a linear map $\phi : A \to A'$ is a morphism of coalgebras if $\delta' \circ \phi = (\phi \otimes \phi) \circ \delta$ and $\varepsilon' \circ \phi = \varepsilon$. A coalgebra, bialgebra or Hopf algebra (A, δ, ε) will often be abbreviated to A. If A, A' are bialgebras a map $\phi : A \to A'$ is a morphism of bialgebras if it is an algebra and coalgebra morphism. If A, A' are Hopf algebras with antipodes σ, σ' then a morphism of bialgebras $\phi : A \to A'$ automatically has the property that $\sigma' \circ \phi = \phi \circ \sigma$, and such a map ϕ is also called a morphism of Hopf algebras.

Let G be a linear algebraic group over K. Then it is easy to check that $(K[G], m^*, \varepsilon_1)$ is a commutative Hopf algebra over K with antipode i^*. Conversely, suppose that (A, δ, ε) is a commutative Hopf K-algebra with antipode σ and suppose that A is finitely generated and reduced (i.e. is without nilpotent elements). Then we associate with A the set $G = \mathrm{Hom}_{K-\mathrm{alg}}(A, K)$ and regard (G, A) as an affine variety, as in 0.2. We have $m : G \times G \to G$ given by $m(x, y) = (x \otimes y) \circ \delta$, $x, y \in G$, and $i : G \to G$ given by $i(x) = x \circ \sigma$, $x \in G$. Moreover G is a group with multiplication $m : G \times G \to G$, inversion $i : G \to G$ and identity ε. By construction we have $f \circ m = \delta(f) \in K[G] \otimes K[G]$ and $f \circ i = \sigma(f) \in K[G]$, for $f \in K[G]$. Thus G is an algebraic group and $m^* = \delta$, $i^* = \sigma$. In this way we obtain an equivalence of categories between linear algebraic groups over K and finitely generated, commutative, reduced Hopf algebras over K.

0.8 As well as preparing the way for our transition to quantum groups, the formalism of the previous paragraph provides a convenient language for discussing rationality properties of algebraic groups. So now let k be a subfield of our algebraically closed field K. For a K-vector space N, a *k-form* of N is a k-subspace M such that the natural map $K \otimes_k M \to N$ is an isomorphism. (This amounts to saying that some, and hence every, k-basis of M is a K-basis of N.) For a K-algebra S we say that R is an algebra k-form of S if R is a k-subalgebra of S and a (vector space) k-form of S. Let $(V, K[V])$ be an affine algebraic variety over K. By the expression "V is a *k-variety*" we indicate that we have in mind a fixed algebra k-form $k[V]$ of $K[V]$. For example, affine n-space $\mathbb{A}^n$ has coordinate algebra $K[X_1, \ldots, X_n]$ and is usually regarded as a k-variety by taking $k[\mathbb{A}^n] = k[X_1, \ldots, X_n]$. Let Z be a closed subset in the affine k-variety V. We say that Z is defined over k if the ideal I_Z is spanned, over K, by $k[V] \cap I_Z$. If this is the case then the natural map $K \otimes_k k[V] \to K[V]$ restricts to an isomorphism $K \otimes_k (k[V] \cap I_Z) \to I_Z$ and Z is viewed as a k-variety with $k[Z]$ consisting of the k-algebra of functions $f|_Z$, for $f \in k[V]$.

Let (C,δ,ε) be a coalgebra over K. We say that B is a coalgebra k-form of C if B is a vector space k-form of C which is closed under the structure maps, i.e. if $\delta(B) \le B \otimes_k B$ and $\varepsilon(B) \le k$. If B is a coalgebra k-form of C then B is naturally a k-coalgebra whose structure maps $B \to B \otimes_k B$ and $B \to k$ are the restrictions of δ and ϵ. If (C,δ,ε) is a bialgebra we say that B is a bialgebra k-form of C if B is both an algebra and coalgebra k-form of C. Finally, if (C,δ,ε) is a Hopf algebra with antipode σ then we say that B is a Hopf k-form of C if B is a bialgebra k-form and $\sigma(B) \le B$. If B is a bialgebra k-form of the K-bialgebra C then B is naturally a bialgebra over k with comultiplication and counit as above. If B is a Hopf k-form of the Hopf K-algebra C, with antipode σ, then B is a Hopf algebra whose antipode is the restriction of σ.

By the expression "G is a *k-group*" or "G is a *linear algebraic group defined over k*" we indicate that G is a linear algebraic group over K and that we have in mind a Hopf k-form of $K[G]$, which we denote $k[G]$. We say that a closed subgroup H is defined over k if H is defined over k, as a closed set in G. In this case H has a natural k-group structure.

Suppose that G and H are k-groups. A map $\phi : G \to H$ is a morphism of k-groups if it is morphism of linear algebraic groups such that $\phi^*(k[H]) \le k[G]$. Thus ϕ gives rise to a morphism of Hopf algebras $k[H] \to k[G]$ and, conversely, a morphism of Hopf algebras $k[H] \to k[G]$ corresponds to a unique morphism of k-groups $G \to H$.

We regard the linear algebraic group $\mathrm{GL}_n(K)$ as a k-group via the Hopf form $k[\mathrm{GL}_n(K)] = k[c_{11}, c_{12}, \ldots, c_{nn}, d^{-1}]$ of $K[\mathrm{GL}_n(K)]$. From now on we shall also write GL_n for $\mathrm{GL}_n(K)$, regarded as a k-group.

0.9 Let (C,δ,ε) be a k-coalgebra. By a right *C-comodule* we mean a pair (V,τ) consisting of a k-space V and a linear map $\tau : V \to V \otimes C$ such that

$$(\tau \otimes \mathrm{id}) \circ \tau = (\mathrm{id} \otimes \delta) \circ \tau : V \to V \otimes C \otimes C$$

and

$$(\mathrm{id} \otimes \varepsilon) \circ \tau = \mathrm{id} : V \to V$$

where id is the identity map on V. We often write simply V for the comodule (V,τ) and call τ the structure map of the comodule V. Let $\{v_i \mid i \in I\}$ be a basis of V. We have elements c_{ij}, $i,j \in I$, defined by the equations

$$\tau(v_i) = \sum_{j \in I} v_j \otimes c_{ji}$$

(for $i \in I$). The k-span of $\{c_{ij} \mid i,j \in I\}$ is called the coefficient space of V and denoted $\mathrm{cf}(V)$. It is independent of the choice of basis of V. Note that $\mathrm{cf}(V)$ is a subcoalgebra of C and that V is naturally a right $\mathrm{cf}(V)$-comodule.

Let (V,τ), (V',τ') be (right) comodules. A linear map $\phi : V \to V'$ is a morphism of comodules (or comodule homomorphism) if $\tau' \circ \phi = (\phi \otimes \mathrm{id}_C) \circ \tau$ (where id_C is the identity map on C). We write $\mathrm{Comod}(C)$ for the category of right C-comodules and $\mathrm{comod}(C)$ for the category of finite dimensional right C-comodules. A subspace U of a comodule V is a subcomodule if $\tau(U) \leq U \otimes C$, where τ is the structure map of V. A subcomodule U is naturally a comodule whose structure map $U \to U \otimes C$ is the restriction of $\tau : V \to V \otimes C$. The structure map τ also induces a map $V/U \to V/U \otimes C$, making V/U into a comodule. The inclusion map $U \to V$ and the quotient map $V \to V/U$ are homomorphisms of comodules. An important feature of the representation theory of coalgebras is the local finiteness property. If V is any right C-comodule then for every finite dimensional subspace T there is a finite dimensional subcomodule U of V such that $T \leq U$. The argument is similar to that of 0.5. Left comodules are defined similarly and have similar properties.

The dual space $C^* = \mathrm{Hom}_k(C,k)$ has the structure of an associative k-algebra with multiplication $\alpha\beta = (\alpha \otimes \beta)\delta$, for $\alpha, \beta \in C^*$, and identity $1_{C^*} = \varepsilon$. For an algebra S, we write $\mathrm{Mod}(S)$ for the category of left S-modules and $\mathrm{mod}(S)$ for the category of finite dimensional left S-modules. Let (V,τ) be a right C-comodule. We regard the k-space V as a left C^*-module via the product $\alpha v = (\mathrm{id} \otimes \alpha)\tau(v)$, for $\alpha \in C^*$, $v \in V$. For $X, Y \in \mathrm{Comod}(C)$, a linear map $\phi : X \to Y$ is a morphism of C-comodules if and only if it is a morphism of C^*-modules. In this way we obtain a full embedding of $\mathrm{Comod}(C)$ into $\mathrm{Mod}(C^*)$. If C is finite dimensional then we obtain, in this way, equivalences of categories $\mathrm{Comod}(C) \to \mathrm{Mod}(C^*)$ and $\mathrm{comod}(C) \to \mathrm{mod}(C^*)$.

0.10 We now return to our discussion of the representation theory of linear algebraic groups in general, and general linear groups in particular. Let G be a linear algebraic group over K. Let V be a (possibly infinite dimensional) rational representation and let $\{v_i \mid i \in I\}$ be a basis of V. We have coefficient functions $f_{ij} \in K[G]$ defined by

$$xv_i = \sum_{j \in I} f_{ji}(x) v_j$$

for $i \in I$. We define $\tau : V \to V \otimes K[G]$ by

$$\tau(v_i) = \sum_{j \in I} v_j \otimes f_{ji}$$

for $i \in I$. Then (V,τ) is a $K[G]$-comodule and we obtain, in this way, an equivalence of categories between rational left G-modules and right $K[G]$-comodules. Similar remarks apply to rational right G-modules and left $K[G]$-comodules.

We identify the categories of left rational G-modules and right $K[G]$-comodules via the above. We now want to discuss the representation theory of k-groups. Let G be a k-group. We shall simply define a left module for G to be a right $k[G]$-comodule and write $\mathrm{Mod}(G)$ (resp. $\mathrm{mod}(G)$) for $\mathrm{Comod}(k[G])$ (resp. $\mathrm{comod}(k[G])$. One could instead decree that a rational G-module over k is a rational G-module V together with a vector space k-form U of V such that for some (and hence every) basis $\{u_i \mid i \in I\}$ the corresponding coefficient functions f_{ij} all belong to $k[G]$. We leave it to the reader to check the equivalence of this with the comodule point of view.

0.11 Let G be a k-group. We write $G(k)$ for the set of $x \in G$ such that $\varepsilon_x(k[G]) \leq k$. It is easy to check that $G(k)$ is a subgroup of G (called the subgroup of *k-rational points*). If $G(k)$ is dense in the Zariski topology on G then we have available yet another description of the category of G-modules (defined over k). This is very close to our original formulation of the notion of a rational module for a linear algebraic group. For $f \in k[G]$ let $\tilde{f}$ be the k-valued function on $G(k)$ defined by $\tilde{f}(x) = \varepsilon_x(f)$, $x \in G(k)$. Let R be the algebra of k-valued functions on $G(k)$ consisting of all functions $\tilde{f}$, with $f \in k[G]$. Note that if $\tilde{f} = 0$ then the closed set $Z = \{x \in G \mid f(x) = 0\}$ of G contains $G(k)$, hence G, and therefore $f = 0$. Thus the map $k[G] \to R$, taking f to $\tilde{f}$, is injective, and we identify $k[G]$ with a k-algebra of functions on $G(k)$ by this map. We say that a finite dimensional $kG(k)$-module V is rational if for some (and hence every) basis $v_1, \ldots, v_n$, the corresponding k-valued functions f_{ij} on $G(k)$, defined by the equations

$$xv_i = \sum_{j=1}^{n} f_{ji}(x)v_j$$

for $1 \leq i \leq n$, all belong to $k[G]$. Given such a $kG(k)$-module we obtain a $k[G]$-comodule structure on V by defining $\tau : V \to V \otimes k[G]$ by $\tau(v_i) = \sum_{j=1}^{n} v_j \otimes f_{ji}$. We leave it to the reader to check that one obtains, in this way, an equivalence of categories between finite dimensional rational left $kG(k)$-modules and finite dimensional right $k[G]$-comodules, hence G-modules (defined over k).

0.12 We consider now the representation theory of a torus, i.e. a linear algebraic group of the form $\mathrm{GL}_1(K)^n = \mathrm{GL}_1(K) \times \cdots \times \mathrm{GL}_1(K)$ (n times). Thus $K[\mathrm{GL}_1(K)^n]$ is the Laurent polynomial algebra $K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$, where $t_i(x_1, \ldots, x_n) = x_i$, $1 \leq i \leq n$, for $(x_1, \ldots, x_n) \in \mathrm{GL}_1(K)^n$. We consider $\mathrm{GL}_1(K)^n$ as a k-group, which we now write as GL_1^n, via the form $k[\mathrm{GL}_1^n] = k[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. We put $X(n) = \mathbb{Z}^n$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in X(n)$ put $t^\alpha = t_1^{\alpha_1} \ldots t_n^{\alpha_n}$. We have the 1-dimensional GL_1^n-module k_α with structure map $k_\alpha \to k_\alpha \otimes k[\mathrm{GL}_1^n]$ taking $x \in k_\alpha$ to $x \otimes t^\alpha$. We have the comodule decomposition $k[\mathrm{GL}_1^n] = \bigoplus_{\alpha \in X(n)} kt^\alpha$. It follows that $\{k_\alpha \mid \alpha \in X(n)\}$ is a complete set of pairwise non-isomorphic irreducible GL_1^n-modules and that every GL_1^n-module is completely reducible.

We form the integral group ring $\mathbb{Z}X(n)$. This has $\mathbb{Z}$-basis $\{e(\alpha) \mid \alpha \in X(n)\}$ whose elements multiply according to the rule $e(\alpha)e(\beta) = e(\alpha+\beta)$. For a GL_1^n-module V and $\alpha \in X(n)$ we have the corresponding weight space $V^\alpha = \{v \in V \mid \tau(v) = v \otimes t^\alpha\}$, where $\tau : V \to V \otimes k[\mathrm{GL}_1^n]$ is the structure map. We say that $\alpha \in X(n)$ is a weight of V if $V^\alpha \neq 0$. The character $\mathrm{ch}\, V$ of a finite dimensional GL_1^n-module is defined by $\mathrm{ch}\, V = \sum_{\alpha \in X(n)} (\dim V^\alpha) e(\alpha) \in \mathbb{Z}X(n)$.

Note that each $\alpha = (\alpha_1, \ldots, \alpha_m) \in X(n)$ gives rise to an algebraic group homomorphism $\mathrm{GL}_1^n \to \mathrm{GL}_1$, also denoted α and given by the formula $\alpha(x) = x_1^{\alpha_1} \ldots x_m^{\alpha_m}$, for $x = (x_1, \ldots, x_m) \in \mathrm{GL}_1^n$. If k is infinite then $\mathrm{GL}_1(k)^n$ is dense in $\mathrm{GL}_1(K)^n$ and, by the formalism of 0.11, for a rational GL_1^n-module V (over K) and $\alpha \in X(n)$ we have $V^\alpha = \{v \in V \mid xv = \alpha(x)v \text{ for all } x \in \mathrm{GL}_1^n\}$.

0.13 We now wish to discuss the rational representation theory of GL_n. Let k be a subfield of K. We view GL_n as a group over k, as in 0.8.

Let $\phi : B \to C$ be a morphism of k-coalgebras. If (V, τ) is a B-comodule then we may regard V as a C-comodule via the structure map $(\mathrm{id} \otimes \phi) \circ \tau : V \to V \otimes C$. We say that this C-comodule is obtained by *ϕ-inflation*, or just inflation if ϕ is inclusion. Suppose that B is a subcoalgebra of C. We say that a C-comodule V *belongs* to B if $\mathrm{cf}(V) \le B$. The C-comodules belonging to B are the objects of a full subcategory of $\mathrm{Comod}(C)$ and inflation defines an equivalence of categories between B-comodules and C-comodules belonging to B.

Let A be a bialgebra. For right comodules $(V, \tau_V), (W, \tau_W)$ we have the tensor product right comodule $(V \otimes W, \tau_{V \otimes W})$. The structure map $\tau_{V \otimes W}$ is given by $\tau_{V\otimes W}(v \otimes w) = \sum_{i,j} v_i \otimes w_j \otimes f_i g_j$, for $v \in V$, $w \in W$ with $\tau_V(v) = \sum_i v_i \otimes f_i$ and $\tau_W(w) = \sum_j w_j \otimes g_j$. Moreover, if A is a Hopf algebra and (V, τ_V) is a finite dimensional right A-comodule then we have the dual right comodule (V^*, τ_{V^*}). The structure map $\tau^* : V^* \to V^* \otimes A$ may be described as follows. Let $v_1, \ldots, v_r$ be a basis of V and let $\alpha_1, \ldots, \alpha_r$ be the dual basis of V^*. Then we have $\tau^*(\alpha_j) = \sum_{i=1}^n \alpha_i \otimes \sigma(f_{ij})$, where $\tau(v_i) = \sum_{j=1} v_j \otimes f_{ji}$, for $1 \le i \le n$, and where σ is the antipode. In the case in which $A = K[G]$, for a linear algebraic group G over K, these constructions correspond, via the formalism of 0.10, to the usual group action on the tensor product of modules and on the dual of a finite dimensional module. Similar remarks apply to left comodules and right modules.

We set $A(n) = k[c_{ij} \mid 1 \le i,j \le n]$. Then $A(n)$ is a subbialgebra of $k[\mathrm{GL}_n]$ and we say that a GL_n-module V is polynomial if $\mathrm{cf}(V) \le A(n)$, i.e.

if V belongs to $A(n)$. We identify the category of polynomial GL_n-modules with the category of $A(n)$-comodules, via the equivalence above. Thus a finite dimensional $K\mathrm{GL}_n(K)$-module V is polynomial if and only if for some (and hence every) basis $\{v_i \mid i \in I\}$ the corresponding coefficient functions f_{ij}, determined by the equations

$$xv_i = \sum_{j \in I} f_{ji}(x)v_j$$

for $x \in \mathrm{GL}_n(K)$ and $i \in I$, all lie in $K[c_{11}, \ldots, c_{nn}]$. More generally, if k is infinite then $\mathrm{GL}_n(k)$ is dense in $\mathrm{GL}_n(K)$ and, by the formalism of 0.11, we have the same description of polynomial GL_n-modules. That is, a finite dimensional polynomial GL_n-module is is a finite dimensional $k\mathrm{GL}_n(k)$-module V such that for some (and hence every) basis $\{v_i \mid i \in I\}$ the corresponding coefficient functions f_{ij}, determined by the equations

$$xv_i = \sum_{j \in I} f_{ji}(x)v_j$$

for $x \in \mathrm{GL}_n(k)$ and $i \in I$, all lie in $k[c_{11}, \ldots, c_{nn}]$. In [**51**], Green considers polynomial GL_n-modules (over an infinite field k) from this perspective.

We have the natural GL_n-module E with basis $e_1, \ldots, e_n$ and structure map $\tau : E \to E \otimes k[G]$ given by $\tau(e_i) = \sum_{j=1}^{n} e_j \otimes c_{ji}$, for $1 \leq i \leq n$. Note that E is a polynomial module. Moreover (since $A(n)$ is a bialgebra) $V \otimes W$ is a polynomial module if V and W are polynomial modules. Hence the rth tensor power $E^{\otimes r}$, the rth symmetric power $S^r E$ and the rth exterior power $\bigwedge^r E$ are polynomial modules, for r any non-negative integer. In particular the determinant module $D = \bigwedge^n E$ is polynomial.

From the point of view of algebraic group representation theory, one is interested in the category of rational modules. However, as we now describe, this differs only trivially from the category of polynomial modules, and the latter category has strong connections with the theory of representations of certain finite dimensional algebras. So let V be a finite dimensional GL_n-module. Then $\mathrm{cf}(V)$ is a finite dimensional subspace of $k[G]$, which is the localization of $A(n)$ at the determinant d. Hence we have $\mathrm{cf}(V) \leq d^{-r}.N$, for some finite dimensional subspace N of $A(n)$, and so $\mathrm{cf}(D^{\otimes r} \otimes V) = \mathrm{cf}(D)^r.\mathrm{cf}(V) = d^r.\mathrm{cf}(V) \leq A(n)$. Thus $D^{\otimes r} \otimes V$ is a polynomial module, for $r \gg 0$, and every finite dimensional rational module is isomorphic to one of the form $D^{\otimes -r} \otimes U$, for some $r \geq 0$ and polynomial module U (where $D^{\otimes -r}$ denotes the dual of $D^{\otimes r}$).

Thus all finite dimensional rational modules can be understood in terms of the polynomial ones. We regard $A(n)$ as a graded algebra by giving each c_{ij} degree 1. So $A(n)$ decomposes

$$A(n) = \bigoplus_{r=0}^{\infty} A(n,r) \qquad (*)$$