This book is an introduction to level set methods and Fast Marching Methods, which are powerful numerical techniques for analyzing and computing interface motion in a host of settings. They rely on a fundamental shift in how one views moving boundaries, rethinking the natural geometric Lagrangian perspective and replacing it with an Eulerian partial differential equation perspectives. The resulting numerical techniques are used to track three-dimensional fronts that can develop sharp corners and change topology as they evolve.

The book begins with an overview of the two techniques, and then provides an introduction to the dynamics of moving curves and surfaces. Next, efficient computational techniques for approximating viscosity solutions to partial differential equations are developed, using numerical technology from hyperbolic conservation laws. This builds a framework for optimal implementations of adaptive techniques for Narrow Band Level Set Methods and Fast Marching Methods. The entire methodology is then redeveloped on triangulated meshes, followed by a series of extensions of the basic ideas. A large collection of applications is given, including examples from physics, chemistry, fluid mechanics, combustion, image processing, materials science, fabrication of microelectronic components, computer vision, control theory, computational geometry, and computer-aided-design and manufacturing.

This book will be a useful resource for mathematicians, applied scientists, practicing engineers, computer graphics artists, and anyone interested in the evolution of boundaries and interfaces.

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3

Level Set Methods and Fast Marching Methods:

Evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science

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and
Fast Marching Methods

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Preface to the Second Edition

The beginning of this work on Level Set Methods and Fast Marching Methods can be found in the author's dissertation on the theory and numerics of propagating interfaces, under the direction of Alexandre Chorin at the University of California at Berkeley. That work continued through a National Science Foundation (NSF) Postdoctoral Fellowship at the Lawrence Berkeley National Laboratory (LBNL) and the Courant Institute of Mathematical Sciences. As I look back on those years, I am extraordinarily grateful for the opportunity that such support gave me to develop these ideas, free from other burdens and responsibilities.

That work on interface methods and its subsequent development at Berkeley have been supported in part by the Applied Mathematical Sciences section of the Department of Energy through the Mathematics Department at LBNL, by NSF awards through the University of California at Berkeley Mathematics Department, and most recently through the Office of Naval Research. Again, I am grateful for all of this support.

I have had the good fortune to work with many collaborators in the development of these ideas. The time-dependent level set formulation of these ideas on interface motion was co-authored with S. J. Osher, whose trips to Berkeley made for a thoroughly enjoyable collaboration. The Narrow Band Level Set Method was developed jointly with D. Adalsteinsson, as was all the work on etching and deposition in semiconductor manufacturing. The work on medical imaging and shape segmentation was joint with R. Malladi, whose help on devising the Fast Marching Method was invaluable. Finally, the triangulated unstructured versions of the Level Set Method are joint with T. Barth of National Aeronautics and Space Administration (NASA) Ames Research Center.

An early application of these techniques, due to D. Chopp, concerns minimal surfaces and includes the genesis of ideas about narrow banding

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and complex boundary conditions. The work on Level Set Methods for crystal growth and dendritic solidification is joint with J. Strain and capitalizes on his boundary integral formulation of the equations of motion. The realization that level set techniques can be applied to shape recovery is due to R. Malladi; many others have capitalized on his ideas. The application of these techniques to image processing began with the work of L. Alvarez, J. M. Morel, and P. L. Lions and the work of S. Osher and L. Rudin; the work on image processing presented here relies heavily on those contributions. The application of the techniques to problems in combustion and fluid interfaces discussed here is joint work with C. Rhee, J. Zhu and L. Talbot, and the work on adaptive mesh refinement relies on the work of B. Milne.

I am fortunate to have on-going collaborations with very talented colleagues, including M. Popovici of 3DGeo Corporation with whom the work on migration and seismic imaging was developed; R. Kimmel, with whom the work on geodesics and the development of a triangulated Fast Marching Method was developed, as well as robotic navigation and optimal path planning; O. Hald, whose analysis of non-convex Hamiltonians was invaluable; as well as L. Borucki and J. Rey in a wide collection of semiconductor issues. At the same time, I am equally fortunate to have been recently benefited from new collaborations with J. Li, A. Sarti, A. Vladimirsky, J. Wei, A. Wiegmann, and J. Wilkening.

Many other people have contributed to the current state-of-the-art of level set methods, including T. Aslam, M. Barlaud, J. Bence, M. Brewer, J. Bzdil, R. Caflisch, V. Caselles, T. Chan, S. Chen, Y. Chen, R. Deriche, V. Dhir, E. Fatemi, O. Faugeras, R. Fedkiw, E. Harabetian, E. Holm, T. Hou, R. Keriven, H.P. Langtangen, D. Lesselier, A. Litman, B. Merriman, E. Pasch, V. Prasad, S. Ruuth, F. Santosa, P. Smereka, N. Sochen, G. Son, S. Stewart, M. Sussman, H. Zhang, H. Zhao, and L.L. Zheng. Their work has advanced both the theory and practice of level set methods. I would like to thank W. Coughran for suggesting the application of level set methods to semiconductor simulations, A. Neureuther for many helpful discussions on etching and deposition, B. Knight for encouragement in the application of level set methods to fluid interface problems, C. Ritchie and G. Chiang for their insightful suggestions about shape recovery in medical imaging, T. Baker for helpful conversations about grid generation, L. Gray for suggesting the application of level set methods to material sintering, and C. Evans for his valuable comments on the initial manuscript. I also wish to thank the students in Math 273 during the fall of 1995 for their insightful comments, critical reviewing, and careful suggestions.

I am also indebted to the many readers of the first edition who have responded with detailed suggestions, and I have been guided by their comments in writing this new edition. At the risk of repetition, I would like to again thank D. Adalsteinsson, D. Chopp, R. Kimmel, R. Malladi, B. Milne, A. Vladimisky, J. Wilkening and J. Zhu. I am extraordinarily fortunate to have had them as colleagues at the Lawrence Berkeley Laboratory.

Finally, I would like to thank Alan Harvey of Cambridge University Press for his thoughtful suggestions and enthusiasm for this project. His calm hand and unfailing humor have made this a pleasure, and his guidance, wise counsel, and wisdom were invaluable. This second and expanded edition is largely due to his optimism, encouragement, and patience.

Berkeley, California, 1999

Introduction

Propagating interfaces occur in a wide variety of settings, and include ocean waves, burning flames, and material boundaries. Less obvious boundaries are equally important and include shapes against backgrounds, handwritten characters, and iso-intensity contours in images. Furthermore, there are applications not commonly thought of as moving interface problems, including optimal path planning and construction of shortest geodesic paths on surfaces, which can be recast as front propagation problems with significant advantages.

The goal of this book is both to unify these ideas and to design a general framework for modeling the evolution of boundaries. The aim is to provide computational techniques for tracking moving interfaces and to give some hint of the flavor and breadth of applications. The work includes examples from physics, chemistry, fluid mechanics, combustion, image processing, materials sciences, fabrication of microelectronic components, computer vision, control theory, seismology, computer-aided-design, and a collection of other areas. The intended audience includes mathematicians, applied scientists, practicing engineers, computer graphics artists, and anyone interested in the evolution of boundaries and interfaces.

Our perspective comes from a large and rapidly growing body of work which relies on a partial differential equations approach for understanding, analyzing, and computing interface motion. At the core lay two computational techniques: "Fast Marching Methods" and "Level Set Methods". Both exploit a fundamental shift in how one views moving boundaries. They rethink the Lagrangian geometric perspective and replace it with an Eulerian, partial differential equation. Fast Marching Methods result from a boundary value problem for the evolving interface, while Level Set Methods result from an associated initial value problem.

In both cases, several advantages stem from this view of propagating interfaces:

- First, from a theoretical/mathematical point of view, some complexities of front motion are illuminated, in particular, the role of singularities, weak solutions, shock formation, entropy conditions, and topological change in the evolving interface.
- Second, from a numerical perspective, natural and accurate ways of computing delicate quantities emerge, including the ability to build high order advection schemes, compute local curvature in two and three dimensions, track sharp corners and cusps, and handle subtle topological changes of merger and breakage.
- Third, from an implementation point of view, since the approaches are based on underlying partial differential equations, robust schemes result from numerical parameters set at the beginning of the computation. The error is thus controlled by
    - (i) the order of the numerical method,
    - (ii) the grid spacing Δh,
    - (iii) in the case of Level Set Methods, the time step Δt; no such requirement exists for Fast Marching Methods.
- Fourth, computational adaptivity is the key to these techniques. In the case of Level Set Methods, the most efficient and preferred approach is the "Narrow Band Level Set Method", which focuses computational labor around the evolving boundary. In the case of Fast Marching Methods, use of standard sorting techniques yields extraordinarily fast and optimally efficient algorithms. In both cases, a clear path to parallelism is available.

This book surveys what is intended to be an illustrative subset of past and current applications of these techniques. We do not assume that the reader is familiar with all of the details required to develop these schemes; the aim is to include the necessary theory and details to provide implementation guidelines.

The first edition of this book was entitled Level Set Methods. The augmented title Level Set Methods and Fast Marching Methods of this new edition embraces the large landscape shared by these two techniques in framing, illuminating, and solving problems with evolving boundaries.

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Outline

This book is divided into four parts. Part I focuses on the formulation of the boundary value and initial value partial differential equations which comprise our two views of interface motion. Part II introduces the theory and numerics underlying Fast Marching Methods and Level Set Methods. Part III introduces the adaptive issues required to construct efficient schemes and variations on the fundamental techniques. Finally, Part IV surveys some application areas.

In **Part I**, Chapter 1 begins with the underlying boundary value and initial value partial differential equations perspective on moving interfaces, and discusses the theoretical and computational advantages of these approaches. It ends with a preview of the rest of the book and provides an outline of the interconnection of the techniques, the relevant theory, numerics, and application strategies. This "look ahead" is meant to provide a structure for the remainder of the book, directing the interested reader to various components of the methodologies.

Part II begins in Chapter 2 with a general statement of the problem of a moving interface and discusses the mathematical theory of curve/surface motion, including the growth/decay of total variation, singularity development, entropy conditions, weak solutions, and shocks in the dynamics of moving fronts. This material has been developed in a collection of papers that are referred to in the text. The viscosity theory of Hamilton-Jacobi equations, which buttresses both computational techniques, is briefly surveyed in Chapter 3.

Chapters 4, 5, and 6 present numerical results which lead up to the Fast Marching and Level Set techniques. Chapter 4 begins with an overview of traditional methods for tracking interfaces, including string methods and cell methods, and makes a first attempt at solving a partial differential equation for front propagation The failure of this first attempt stems from the relationship between front propagation and hyperbolic conservation laws and is the subject of Chapter 5. Chapter 6 then provides a detailed description of straightforward (though inefficient) algorithms for solving the initial value and boundary value problems.

Part III provides complete details on state-of-the-art Fast Marching and Level Set algorithms. It begins in Chapter 7 with a discussion of computational adaptivity. After surveying work on parallel and adaptive mesh approaches, the chapter focuses on the Narrow Band Level Set Method. This is the most efficient and accurate way to implement level set methods. Next, in Chapter 8, Fast Marching Methods are introduced, which are the optimal way to solve Hamilton-Jacobi equations which arise from certain interface motion problems. The techniques require a detailed discussion of causality in upwind schemes and optimal heap sort algorithms. Higher accuracy versions of both Narrow Band and Fast Marching Methods are supplied.

Next, in Chapters 9 and 10, the entire framework is moved to a triangulated unstructured mesh setting. Schemes for the Level Set Method are given, including monotone schemes, positive schemes, Petrov-Galerkin schemes, as well as explicit and implicit schemes with discontinuity capturing. In the case of Fast Marching Methods, upwind causality schemes for both acute and non-acute triangulations are introduced. These two sets of schemes provide versatile techniques for interface propagation problems on manifolds and in irregular domains.

Chapter 11 explains how to build general level set methods in many physical problems. It examines how to build appropriate and natural methods for moving the neighboring level sets, which is required in order to implement level set techniques. Detailed techniques for generating smooth level set flows which avoids all re-initialization are given, as are techniques for obtaining sub-grid accuracy. In Chapter 12, the numerical accuracy and robustness tests are measured, including scheme convergence rates, tests of triangulated techniques, examination of mass conservation and accuracy. Finally, in Chapter 13, the underlying philosophy of Narrow Band and Fast Marching Methods applications is discussed.

Part IV focuses on applications of both the Narrow Band Level Set Method and Fast Marching Methods to a collection of problems. Here, the intent is to show the breadth of current applications and to serve as a guidepost for further research. Chapter 14 begins with some pure geometry problems, including curve/surface shrinkage, the existence of self-similar surfaces, flows under more complex metrics, sintering and second derivative of curvature flows, triple points, multiple interfaces, and constraint-based flows. Chapter 15 extends this work and shows how these techniques can be used in grid generation, giving many examples of body-fitted logical rectangular grids around complex bodies in two and three dimensions. Chapter 16 moves to image processing and views images as collections of iso-intensity contours; by constructing a suitable speed law, these contours can be allowed to propagate in a way that both removes noise and enhances desired regions.

Chapter 17 focuses on aspects of computer vision. It begins with the problem of shape-from-shading and then shows how to transform image segmentation problems into moving interface versions of active contours; when driven by gradients in the image field, these contours extract desired shapes from images. Applications are drawn from a wide collection of medical data, including three-dimensional scans of cortical and cardiac structures. Once images are segmented, the next step is recognition, which is discussed in the context of automatic identification of meteorological data and optical character recognition.

Chapter 18 provides examples of interface problems in which the physics on each side of the interface both drives the front and is affected by the front location and properties. Several areas are discussed, including combustion and flame propagation, crystal growth and dendritic solidification, fluid interface transport and two-phase flow, and electromigration. In all four, the front is driven both by local effects and by underlying transport terms. General guidelines for arbitrary fluid/material interface problems are given. Additional applications include boiling, groundwater transport, and liquid bridges.

Chapter 19 focuses on various aspects of computational geometry and computer-aided-design. It begins with efficient algorithms for shape-offsetting. Next, techniques for constructing minimal surfaces are given which rely on constrained fronts evolving under mean curvature until final minimal steady states are achieved. The chapter ends with problems of shape smoothing, of importance in removing noise from range images as well as machine part manufacturing. This is performed using variants of the image smoothing schemes presented earlier.

Chapter 20 applies Fast Marching methods to a variety of problems in computing first arrivals, optimization, and control. It begins with problems in path planning and navigation under constraints. It then gives an optimal algorithm for constructing shortest path geodesics on complex triangulated manifolds, including a technique for ruling surfaces. It then discusses first arrival times of seismic waves in geophysical migration modeling and problems applying level set methods to air-traffic control. The chapter ends with some new algorithms for computing visibility in complex scenes.

Chapter 21 presents the most sophisticated application of Fast Marching/Level Set methodologies to date, namely the simulation of etching, deposition, and photolithography development in the microfabrication of semiconductor components. Here, photolithography development in planar and non-planar domains, etching and deposition with non-convex ion-milling sputter effects, re-emission and re-deposition mechanisms

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with small sticking coefficients, passive sidewall activation, surface diffusion and re-flow, as well as full three-dimensional effects are discussed. This requires the use of efficient visibility schemes, schemes for sintering, fast solution of flux integral equations, and sub-grid adaptivity schemes.

By no means is this an exhaustive review of the work that exists on Fast Marching Methods and Level Set Methods. A body of work has been reluctantly skipped in the effort to keep this book to reasonable length. The interested reader is referred to a wide range of simulations developed using these methodologies; references will be given throughout the text. The goal of this book is to provide windows into these techniques as guides for further interface studies.

The author can be reached at sethian@math.berkeley.edu. A general article on Fast Marching Methods and Level Set Methods may be found in [237], other reviews may be found in [238] and [235]. Finally, a web page devoted to the topic of Fast Marching Methods and Level Set Methods may be found at http://math.berkeley.edu/~sethian/level_set.html.