Part I

Equations of Motion for Moving Interfaces

Part I presents the underlying partial differential equations perspective on moving interfaces. One view leads to a boundary value partial differential equation for the evolving front, the other leads to a time-dependent initial value problem. The goal is to lay out clearly the two views and discuss the theoretical and computational advantage of these approaches.

Cambridge University Press
978-0-521-64557-7 - Level Set Methods and Fast Marching Methods: Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision, and Materials Science
J. A. Sethian
Excerpt
More information

1
Formulation of Interface Propagation

Outline: *We formulate the boundary value and initial value partial differential equations which describe interface motion. These will eventually lead to the Fast Marching Method and the Narrow Band Level Set Method; for now, however, we focus on the theoretical and computational advantages that come from these perspectives.*

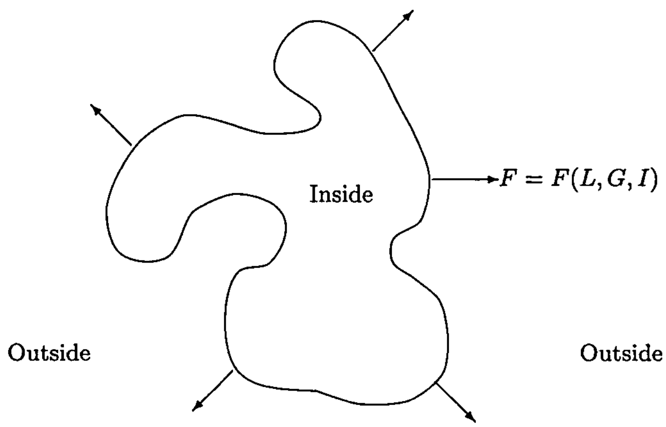


Fig. 1.1. Curve propagating with speed F in normal direction.

Consider a boundary, either a curve in two dimensions or a surface in three dimensions, separating one region from another. Imagine that this curve/surface moves in a direction normal to itself (where the normal direction is oriented with respect to an inside and an outside) with a known speed function F. The goal is to track the motion of this interface as it evolves. We are concerned only with the motion of the interface in its normal direction; throughout, we shall ignore motions of the interface in its tangential directions.

The speed function F, which may depend on many factors, can be written as:

$$F = F(L, G, I), \tag{1.1}$$

where

- *L= Local properties* are those determined by local geometric information, such as curvature and normal direction.
- *G= Global properties of the front* are those that depend on the shape and position of the front. For example, the speed might depend on integrals along the front and/or associated differential equations. As a particular case, if the interface is a source of heat that affects diffusion on either side of the interface, and a jump in the diffusion in turn influences the motion of the interface, then this would be characterized as global property.
- *I= Independent properties* are those that are independent of the shape of the front, such as an underlying fluid velocity that passively transports the front.

Much of the challenge in interface problems comes from producing an adequate model for the speed function F; this is a separate issue independent of the goal of an accurate scheme for advancing the interface based on the model for F. In this chapter, it is assumed that the speed function F is known. The goal of Part IV is to formulate good models for F for a collection of applications.

Given F and the position of an interface, the objective is to track the evolution of the interface. Our first task is to formulate this evolution problem in an *Eulerian* framework, that is, one in which the underlying coordinate system remains fixed.

1.1 A boundary value formulation

Assume for the moment that $F > 0$, hence the front always moves "outward." One way to characterize the position of this expanding front is to compute the arrival time $T(x, y)$ of the front as it crosses each point (x, y), as shown in Figure 1.2.

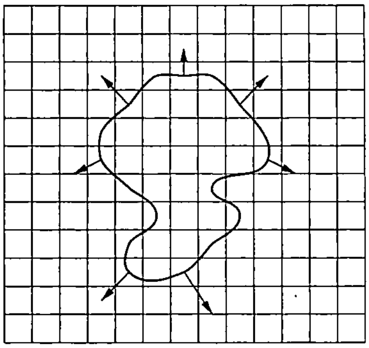


Fig. 1.2. Calculation of crossing time at (x, y) for expanding front $F > 0$.

The equation for this arrival function $T(x, y)$ is easily derived. In one dimension, using the fact that *distance* = *rate* ∗ *time* (see Figure 1.3), we have that

$$1 = F\,\frac{dT}{dx}.$$

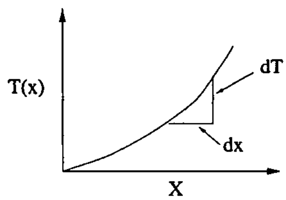


Fig. 1.3. Setup for boundary value formulation.

In multiple dimensions, ∇T is orthogonal to the level sets of T, and, similar to the one-dimensional case, its magnitude is inversely proportional to the speed. Hence

$$|\nabla T| F = 1, \qquad T = 0 \text{ on } \Gamma, \tag{1.2}$$

where Γ is the initial location of the interface.

Thus, the front motion is characterized as the solution to a boundary value problem. If the speed F depends only on position, then the equation reduces to what is known as the "Eikonal" equation. As an example, the arrival surface $T(x, y)$ for a circular front expanding with unit speed $F = 1$ is shown in Figure 1.4.

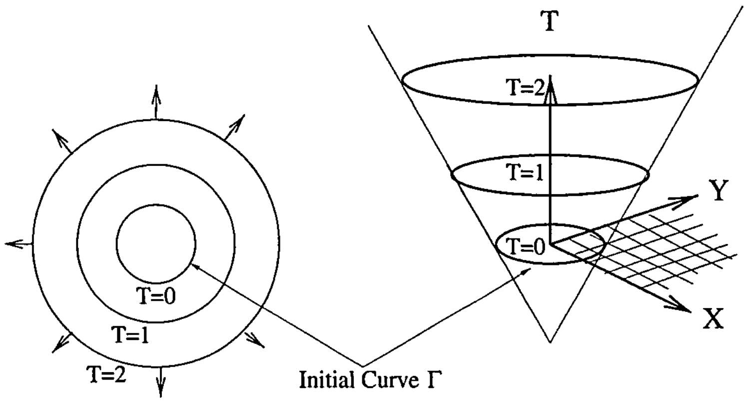


Fig. 1.4. Transformation of front motion into boundary value problem.

1.2 An initial value formulation

Conversely, suppose now that the front moves with a speed F that is neither strictly positive nor negative. Then we must account for the fact that the front can move forward and backward, and hence can pass over a point (x, y) several times. Thus, the crossing time $T(x, y)$ is not a single-valued function. Our way of taking care of this is to embed the initial position of the front as the zero level set of a higher-dimensional function ϕ. We can then link the evolution of this function ϕ to the propagation of the front itself through a time-dependent initial value problem. At any time, the front is given by the zero level set of the time-dependent level set function ϕ (see Figure 1.5).

In order to derive an equation of the motion for this level set function ϕ and match the zero level set of ϕ with the evolving front, we first require that the level set value of a particle on the front with path $x(t)$ must always be zero, and hence

$$\phi(x(t), t) = 0. \tag{1.3}$$

By the chain rule,

$$\phi_t + \nabla\phi(x(t), t) \cdot x'(t) = 0. \tag{1.4}$$

Since F supplies the speed in the outward normal direction, then $x'(t) \cdot n = F$, where $n = \nabla\phi / |\nabla\phi|$. This yields an evolution equation for ϕ, namely

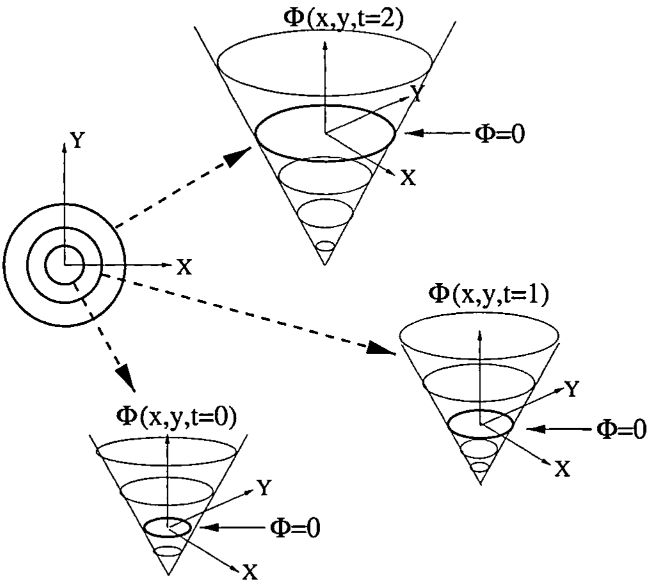


Fig. 1.5. Transformation of front motion into initial value problem.

$$\phi_t + F|\nabla\phi| = 0, \tag{1.5}$$

$$\text{given} \quad \phi(x, t = 0).$$

This is the level set equation given by Osher and Sethian [187]. For certain forms of the speed function F, one obtains a standard Hamilton–Jacobi equation. Equation 1.5 describes the time evolution of the level set function ϕ in such a way that the zero level set of this evolving function is always identified with the propagating interface; see Figure 1.5.

Thus, we can summarize our two perspectives. Let Γ be a curve in the plane propagating in a direction normal to itself with speed F such that $\Gamma(t)$ gives the position of the front at time t. Then, we wish to solve

Boundary Value Formulation	**Initial Value Formulation**
$\lvert\nabla T\rvert F = 1$	$\phi_t + F\lvert\nabla\phi\rvert = 0$
Front= $\Gamma(t) = \{(x,y) \mid T(x,y) = t\}$	Front=$\Gamma(t) = \{(x,y) \mid \phi(x,y,t) = 0\}$
Requires $F > 0$	Applies for arbitrary F

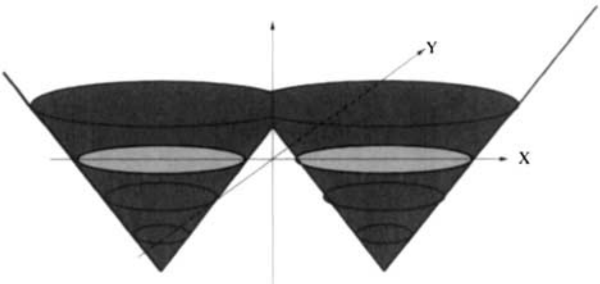
(1.6)

1.3 Advantages of these perspectives

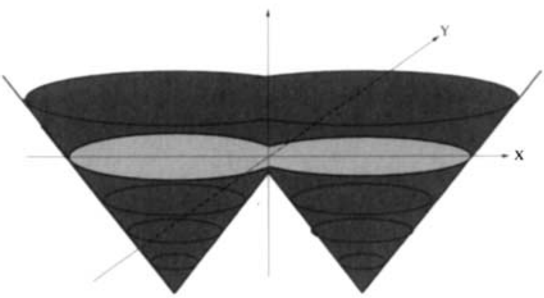
There are certain advantages associated with these two perspectives on propagating interfaces.

- Both are unchanged in higher dimensions, that is, for hypersurfaces propagating in three dimensions and higher.

- Topological changes in the evolving front Γ are handled naturally. The position of the front at time t is given either by the zero level set $\phi(x, y, t) = 0$ of the evolving function ϕ or by the level set $T(x, y) = t$ of the boundary value solution. This set need not be a single curve, and it can break and merge as t advances. In both cases, the key fact is that the boundary value solution $T(x, y)$ and the level set function ϕ remain single-valued (see Figure 1.6).



The level set surface ϕ (dark gray):
Two separate initial fronts (in light gray).



Later in time: the interface topology has changed, yielding a single curve as the zero level set.

Fig. 1.6. Topological change.

- Both rely on viscosity solutions of the associated partial differential equations in order to guarantee that the unique, entropy-satisfying weak solution is be obtained.

- Both are accurately approximated by computational schemes which exploit techniques borrowed from the numerical solutions of hyperbolic conservation laws. For example, schemes may be developed by using a discrete grid in x-y domain and substituting finite difference approximations[1] for the spatial and temporal derivatives. As illustration, using a uniform mesh of spacing h, with grid nodes (i,j), and employing the standard notation that ϕ_{ij}^n is the approximation to the solution $\phi(ih, jh, n\Delta t)$, where Δt is the time step, one might write
$$\frac{\phi_{ij}^{n+1} - \phi_{ij}^n}{\Delta t} + F|\nabla_{ij}\phi_{ij}^n| = 0. \tag{1.7}$$
Here, a forward difference scheme in time has been used, and $|\nabla_{ij}\phi_{ij}^n|$ represents some appropriate finite difference operator for the spatial derivative. Thus, an explicit finite difference approach is possible. The construction of correct entropy-satisfying approximations to the difference operator is the subject of Part II; for now, the important fact is that one has an explicit error control on the basis of the initial spatial discretization and the order of the numerical scheme.

- Intrinsic geometric properties of the front are easily determined in both formulations. For example, at any point of the front, the normal vector is given by
$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} \quad \text{or} \quad \vec{n} = \frac{\nabla T}{|\nabla T|}, \tag{1.8}$$
and the curvature of the front at any point is easily obtained from the divergence of the unit normal vector to the front, i.e.,
$$\kappa = \left\{ \begin{array}{lcl} \nabla \cdot \frac{\nabla\phi}{|\nabla\phi|} & = & \frac{\phi_{xx}\phi_y^2 - 2\phi_x\phi_y\phi_{xy} + \phi_{yy}\phi_x^2}{(\phi_x^2+\phi_y^2)^{3/2}} \\ \nabla \cdot \frac{\nabla T}{|\nabla T|} & = & \frac{T_{xx}T_y^2 - 2T_xT_yT_{xy} + T_{yy}T_x^2}{(T_x^2+T_y^2)^{3/2}} \end{array} \right\}. \tag{1.9}$$

- Both methods are made efficient through the use of adaptive computational strategies, which lead to Narrow Band Level Set Methods and Fast Marching Methods.

[1] Finite difference approximations will be discussed in detail in Chapter 5.

At the same time, there are significant differences between the two approaches.

- The most obvious difference is that the initial value level set formulation allows for both positive and negative speed functions F; the front may move forward and backward as it evolves. The boundary value perspective is restricted to fronts that always move in the same direction, because it requires a single crossing time T at each grid point, and hence a point cannot be revisited. Thus, models involving more complex speed functions F, such as those including curvature, are most naturally framed as initial value level set problems.
- Conversely, positive speed functions F which depend on position and vary widely from point to point are best framed as boundary value problems and approximated through the use of Fast Marching Methods. This is because

  (i) The boundary value formulation requires no time step, and hence its approximation is not subject to CFL conditions, unlike Level Set Methods.

  (ii) Through the use of heap sort algorithms, Fast Marching Methods can be made extremely computationally efficient, far eclipsing Level Set Methods.

1.4 A general framework

We can be slightly informal and describe both formulations with the general partial differential equation

$$\alpha u_t + H(Du, x) = 0. \tag{1.10}$$

Here, Du represents the partials of u in each variable, for example, u_x and u_y. In the case of the Eikonal equation, $\alpha = 0$, and the function H reduces to $H = F|\nabla u| - 1$.

One of the main subtleties that arises in solving this equation is that the solution need not be differentiable, even with arbitrarily smooth boundary data. This non-differentiability is intimately connected to the notion of appropriate weak solutions. Our goal will be to construct numerical techniques which naturally account for this non-differentiability in the construction of accurate and efficient approximation schemes and admit physically correct non-smooth solutions.

1.5 A look ahead/A look back

It is worthwhile to stop and explain how these techniques were developed and what lies ahead. The first step in the development of these ideas started with the analysis of corners and singularities in propagating interfaces. In [222, 225], the role of curvature as a regularizing or smoothing term was investigated, and it was shown that this regularizing role connects to the notion of entropy conditions and shocks in hyperbolic conservation laws in gas dynamics. This is the subject of Chapter 2. A more formal view comes from considering viscosity solutions of Hamilton-Jacobi equations, which is the subject of Chapter 3.

The second step in the development of accurate and efficient numerical techniques for interface evolution comes from the realization that the schemes from computational fluid mechanics, specifically designed for approximating the solution to hyperbolic conservation laws, can be used to solve the equations of front propagation. This was the view developed in [226], and is at the core of modern interface methods:

> "Most algorithms place marker particles along the front and advance the position of the particles in accordance with a set of finite difference approximations to the equations of motion. Such schemes usually go unstable and blow up as the curvature builds around a cusp, since small errors in the position produce large errors in the determination of the curvature. One alternative is to consider the reformulation equations of motion as a conservation law with viscosity and solve these equations with the techniques developed for gas dynamics. These techniques, based on high-order upwind formulations, are particularly attractive, since they are highly stable, accurate and preserve monotonicity. We have made some preliminary tests of such schemes applied to our problem of propagating fronts in crystals and flames, with extremely encouraging results..."

To execute this strategy, we need schemes from hyperbolic conservation laws; this is the subject of Chapters 4 and 5.

The combination of these three subjects then leads to the two numerical schemes given in Chapter 6: the Level Set Method ([187]) for the initial value problem, and an iterative method for the boundary value problem. They are made efficient in Chapters 8 and 9 through adaptivity, leading to Narrow Band Level Set Methods, see [2], and Fast Marching Methods, see [233]. Finally, after a series of extensions of