

## **Probability and Random Variables**

### **A Beginner's Guide**

This is a simple and concise introduction to probability theory. Self-contained and readily accessible, it is written in an informal tutorial style with a humorous undertone. Concepts and techniques are defined and developed as necessary. After an elementary discussion of chance, the central and crucial rules and ideas of probability, including independence and conditioning, are set out. Counting, combinatorics, and the ideas of probability distributions and densities are then introduced. Later chapters present random variables and examine independence, conditioning, covariance, and functions of random variables, both discrete and continuous. The final chapter considers generating functions and applies this concept to practical problems including branching processes, random walks, and the central limit theorem. Examples, demonstrations, and exercises are used throughout to explore the ways in which probability is motivated by, and applied to, real-life problems in science, medicine, gaming, and other subjects of interest. Essential proofs of important results are included.

Since it assumes minimal prior technical knowledge on the part of the reader, this book is suitable for students taking introductory courses in probability and will provide a solid foundation for more advanced courses in probability and statistics. It would also be a valuable reference for those needing a working knowledge of probability theory and will appeal to anyone interested in this endlessly fascinating and entertaining subject.

# Probability and Random Variables

## A Beginner's Guide

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## Synopsis

This is a simple and concise introduction to probability and the theory of probability. It considers some of the ways in which probability is motivated by, and applied to, real-life problems in science, medicine, gaming, and other subjects of interest. Probability is inescapably mathematical in character but, as befits a first course, the book assumes minimal prior technical knowledge on the part of the reader. Concepts and techniques are defined and developed as necessary, making the book as accessible and self-contained as possible.

The text adopts an informal tutorial style, with emphasis on examples, demonstrations, and exercises. Nevertheless, to ensure that the book is appropriate for use as a textbook, essential proofs of important results are included. It is therefore well suited to accompany the usual introductory lecture courses in probability. It is intended to be useful to those who need a working knowledge of the subject in any one of the many fields of application. In addition it will provide a solid foundation for those who continue on to more advanced courses in probability, statistics, and other developments. Finally, it is hoped that the more general reader will find this book useful in exploring the endlessly fascinating and entertaining subject of probability.

On this occasion, I must take notice to such of my readers as are well versed in Vulgar Arithmetic, that it would not be difficult for them to make themselves Masters, not only of all the practical Rules in this book, but also of more useful Discoveries, if they would take the small Pains of being acquainted with the bare Notation of Algebra, which might be done in the hundredth part of the Time that is spent in learning to write Short-hand.

A. de Moivre, *The Doctrine of Chances*, 1717



## Preface

This book begins with an introduction, chapter 1, to the basic ideas and methods of probability that are usually covered in a first course of lectures. The first part of the main text, subtitled Probability, comprising chapters 2–4, introduces the important ideas of probability in a reasonably informal and non-technical way. In particular, calculus is not a prerequisite.

The second part of the main text, subtitled Random Variables, comprising the final three chapters, extends these ideas to a wider range of important and practical applications. In these chapters it is assumed that the student has had some exposure to the small portfolio of ideas introduced in courses labelled ‘calculus’. In any case, to be on the safe side and make the book as self-contained as possible, brief expositions of the necessary results are included at the ends of appropriate chapters.

The material is arranged as follows.

Chapter 1 contains an elementary discussion of what we mean by probability, and how our intuitive knowledge of chance will shape a mathematical theory.

Chapter 2 introduces some notation, and sets out the central and crucial rules and ideas of probability. These include independence and conditioning.

Chapter 3 begins with a brief primer on counting and combinatorics, including binomial coefficients. This is illustrated with examples from the origins of probability, including famous classics such as the gambler’s ruin problem, and others.

Chapter 4 introduces the idea of a probability distribution. At this elementary level the idea of a probability density, and ways of using it, are most easily grasped by analogy with the discrete case. The chapter therefore includes the uniform, normal, and exponential densities, as well as the binomial, geometric, and Poisson distributions. We also discuss the idea of mean and variance.

Chapter 5 introduces the idea of a random variable; we discuss discrete random variables and those with a density. We look at functions of random variables, and at conditional distributions, together with their expected values.

Chapter 6 extends these ideas to several random variables, and explores all the above concepts in this setting. In particular, we look at independence, conditioning, covariance, and functions of several random variables (including sums). As in chapter 5 we treat continuous and discrete random variables together, so that students can learn by the use of analogy (a very powerful learning aid).

Chapter 7 introduces the ideas and techniques of generating functions, in particular probability generating functions and moment generating functions. This ingenious and

elegant concept is applied to a variety of practical problems, including branching processes, random walks, and the central limit theorem.

In general the development of the subject is guided and illustrated by as many examples as could be packed into the text. Nevertheless, I have not shrunk from including proofs wherever they are important, or informative, or entertaining.

Naturally, some parts of the book are easier than others, and I would offer readers the advice, which is very far from original, that if they come to a passage that seems too difficult, then they should skip it, and return to it later. In many cases the difficulty will be found to have evaporated.

In general it is much easier and more pleasant to get to grips with a subject if you believe it to be of interest in its own right, rather than just a handy tool. I have therefore included a good deal of background material and illustrative examples to convince the reader that probability is one of the most entertaining and endlessly fascinating branches of mathematics. Furthermore, even in a long lecture course the time that can be devoted to examples and detailed explanations is necessarily limited. I have therefore endeavoured to ensure that the book can be read with a minimum of additional guidance.

Moreover, prerequisites have been kept to a minimum, and mathematical complexities have been rigorously excluded. You do need common sense, practical arithmetic, and some bits of elementary algebra. These are included in the core syllabus of all school mathematics courses.

Readers are strongly encouraged to attempt a respectable fraction of the exercises and problems. Tackling relevant problems (even when the attempt is not completely successful) always helps you to understand the concepts. In general, the exercises provide routine and transparent applications of ideas in the nearby text. Problems are often less routine; they may use ideas from further afield, and may put them in a new setting. Solutions and hints for most of the exercises and problems appear before the Index.

While all the exercises and problems have been kept as simple and straightforward as possible, it is inescapable that some may seem harder than others. I have resisted the temptation to magnify any slight difficulty by advertising it with an asterisk or equivalent decoration. You are at liberty to find any exercise easy, irrespective of any difficulties I may have anticipated.

It is certainly difficult to exclude every error from the text. I entreat readers to inform me of all those they discover.

Finally, you should note that the ends of examples, definitions, and proofs are denoted by the symbols  $\circ$ ,  $\triangle$ , and  $\square$  respectively.

Oxford  
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*Postscript:* 2001. I am extremely grateful to John Scollin of Sydney, Australia, who has read the first printing with unusual care and attention. The errors thereby revealed have all been corrected in this printing, but you should note that the penultimate sentence of the above preface still remains in force.