Introduction to algebraic groups and Lie algebras

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1 Basic concepts

We begin with some basic definitions relating to algebraic groups.

Let k be an algebraically closed field and k^n the vector space of n-tuples over k. An *affine variety* over k is a subset of k^n for some n which is defined by the vanishing of a set of polynomial equations. A *morphism of affine varieties* is a map $\phi : V_1 \to V_2$ such that, for $v \in V_1$, the coordinates of $\phi(v)$ are polynomial functions in the coordinates of v. If $V \subset k^n$ and $V' \subset k^{n'}$ are affine varieties it is clear that $V \times V' \subset k^{n+n'}$ will be an affine variety also.

An *affine algebraic group* over k is a set G which is both an affine variety and a group such that the maps $(x, y) \to xy$ from $G \times G$ to G and $x \to x^{-1}$ from G to G are morphisms of varieties.

For example consider the special linear group given by

$$SL_n(k) = \{(a_{ij}) \in k^{n^2} ; \det(a_{ij}) = 1\}.$$

Then $SL_n(k)$ is an affine algebraic group. Its group structure is given by matrix multiplication and its variety structure is given by the fact that it is the set of points in k^{n^2} given by the vanishing of a single polynomial equation.

We note that the same argument does not apply to the general linear group $GL_n(k)$ given by

$$GL_n(k) = \{(a_{ij}) \in k^{n^2} ; \det(a_{ij}) \neq 0\}.$$

However $GL_n(k)$ can be given the structure of an affine algebraic group by considering it as a subset of k^{n^2+1} instead. We have

$$GL_n(k) = \{(a_{ij}, b) \in k^{n^2+1} ; b \det(a_{ij}) = 1\}.$$

A *homomorphism* of affine algebraic groups is a map $\phi : G_1 \to G_2$ which is a morphism of varieties and a homomorphism of groups. An *isomorphism* of affine algebraic groups is a map $\phi : G_1 \to G_2$ which is bijective such that ϕ and ϕ^{-1} are homomorphisms of affine algebraic groups.

We now consider subvarieties of affine algebraic varieties. If $V \subset k^n$ is an affine variety the subvarieties of V are those subsets of V which are themselves affine varieties in k^n. The subvarieties of V form the closed sets in a topology called the *Zariski topology* on V.

Given an affine algebraic group G it is natural to consider the closed subgroups of G, since such subgroups are themselves affine algebraic groups.

A *linear algebraic group* is a closed subgroup of $GL_n(k)$ for some n. Thus every linear algebraic group is an affine algebraic group. In fact the converse is true also. Given any affine algebraic group G there is an isomorphism of affine algebraic groups between G and a closed subgroup of $GL_n(k)$ for some n. Thus any affine algebraic group is isomorphic to a linear algebraic group. The terms 'affine algebraic group' and 'linear algebraic group' are thus interchangeable. It is more common to use the term 'linear algebraic group'.

We mention three basic reference books on linear algebraic groups where more details and proofs of the results mentioned here can be found. They are:

Linear Algebraic Groups by A. Borel.

Linear Algebraic Groups by J.E. Humphreys.

Linear Algebraic Groups by T.A. Springer.

(See the precise references at the end of this chapter.)

2 Linear algebraic groups

Let G be a linear algebraic group. Then G, considered as an affine variety, will be the union of finitely many irreducible components. As a topological space G will be the disjoint union of its connected components. In fact it can be shown that the irreducible components coincide with the connected components. These will simply be called the components of G. Thus G is the disjoint union of finitely many components. Let G^0 be the component containing the identity element. Then the other components turn out to be simply the cosets of G^0 in G. Furthermore, G^0 is a normal subgroup of G of finite index and the set of cosets G^0x, $x \in G$, is the set of components of G.

An element $x \in G$ is called *semisimple* if x is a diagonalisable matrix, i.e. conjugate in the general linear group to a diagonal matrix. It can be shown that this condition is independent of the matrix representation of G. Thus the property of being semisimple depends only on the linear algebraic group G and not on the particular matrix representation which is used.

An element $x \in G$ is called *unipotent* if all the eigenvalues of the matrix x are equal to 1. This condition is again independent of the matrix representation of G.

It can be shown that each element $x \in G$ is uniquely expressible in the form

$$x = x_s x_u = x_u x_s$$

where x_s is semisimple and x_u is unipotent. This is called the *Jordan decomposition* of x. This decomposition plays a key role in the theory of algebraic groups.

We now define a number of subgroups in a linear algebraic group G.

$R(G)$ is the *radical* of G. This is the unique maximal closed connected soluble normal subgroup of G.

$R_u(G)$ is the *unipotent radical* of G. This is the unique maximal closed connected normal subgroup of G, all of whose elements are unipotent. An algebraic group is called unipotent if all its elements are unipotent. We have

$$R_u(G) \subset R(G).$$

The group G is called *reductive* if $R_u(G) = 1$. It is called *semisimple* if $R(G) = 1$. Thus any semisimple group is reductive, but not conversely. The factor group $G/R_u(G)$ is a reductive group and the factor group $G/R(G)$ is semisimple. In fact the factor group G/N of a linear algebraic group with respect to a closed normal subgroup N has the structure of a linear algebraic group, well defined up to isomorphism.

A linear algebraic group G is called *simple* if G is connected but G has no proper closed connected normal subgroup. Every abelian simple group has dimension 1 (as an affine variety). There are two such groups, the additive group k^+ and the multiplicative group k^*. k^+ can conveniently be defined as

$$k^+ = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} ; \lambda \in k \right\}$$

since

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda + \mu \\ 0 & 1 \end{pmatrix}.$$

The multiplicative group k^* is defined by $k^* = GL_1(k)$ since

$$(\lambda)(\mu) = (\lambda\mu) \qquad \lambda \neq 0, \mu \neq 0.$$

If G is a non-abelian simple algebraic group then G is semisimple. The centre $Z(G)$ of such a group is finite, and the factor group $G/Z(G)$ is simple as an abstract group. G itself need not be simple as an abstract group. For example $SL_2(k)$ is simple as an algebraic group but not necessarily as an abstract group. For this reason the algebraic groups we have defined as simple are sometimes called almost simple groups.

3 Maximal tori and Borel subgroups

It is possible to define the direct product of linear algebraic groups in a natural way. In particular, an algebraic group isomorphic to a direct product

$$k^* \times \cdots \times k^*$$

of multiplicative groups is called a *torus*. A maximal torus in a linear algebraic group G is a closed subgroup of G which is a torus but is not contained in any larger torus. It can be shown that any two maximal tori are conjugate in G.

Let G be a connected linear algebraic group and T be a maximal torus of G. Let $C(T)$ be the centralizer of T and $N(T)$ be the normalizer of T. Then we have

$$T \subset C(T) \subset N(T) .$$

In fact one can show that $C(T) = N(T)^0$, the connected component of $N(T)$ containing 1. In particular $N(T)/C(T)$ is finite. We write

$$W = N(T)/C(T) .$$

W is called the *Weyl group* of G. It is well defined by G up to isomorphism, since any two maximal tori of G are conjugate. In the special case when G is a connected reductive group we have $T = C(T)$. Thus in this case we have

$$W = N(T)/T .$$

The Weyl group W is a finite Coxeter group, i.e. it is a finite group which can be defined by generators and relations of the form

$$W = \langle s_1, \ldots , s_n; \quad \begin{array}{ll} s_i^2 = 1 & \text{for } i = 1, \ldots , n \\ (s_i s_j)^{m_{ij}} = 1 & \text{for } i \neq j \rangle . \end{array}$$

Suppose, for example, that $G = GL_n(k)$. This is a connected reductive group. The subgroup $D_n(k)$ of diagonal matrices is a maximal torus T of G. The normalizer $N(T)$ consists of all monomial matrices in G. Thus we have

$$N(T)/T \cong S_n$$

where S_n is the symmetric group of degree n. Thus the Weyl group W is isomorphic to S_n. Consider the transpositions of consecutive integers given by

$$s_1 = (1,2) , \quad s_2 = (2,3) , \quad \ldots , \quad s_{n-1} = (n-1, n).$$

Then we have

$$
\begin{aligned}
s_i^2 &= 1 && \text{for} \quad i = 1, \dots, n-1. \\
(s_i s_j)^2 &= 1 && \text{for} \quad |i-j| > 1. \\
(s_i s_j)^3 &= 1 && \text{for} \quad |i-j| = 1.
\end{aligned}
$$

Moreover $s_1, \dots, s_{n-1}$ generate $S_n \cong W$ as a Coxeter group.

We now consider a type of subgroup, introduced by A. Borel, which plays a key role in the theory of algebraic groups. A *Borel subgroup* of a linear algebraic group G is a maximal closed connected soluble subgroup of G. (In fact a maximal connected soluble subgroup is always closed.) It can be shown that any two Borel subgroups of G are conjugate. This is proved by showing that, if B is a Borel subgroup of G, then the set G/B of left cosets can be regarded as a projective variety and that a connected soluble group, when acting on a complete variety (in particular on a projective variety) always has a fixed point. This illustrates the fact that in order to develop the theory of linear algebraic groups one cannot remain within the category of affine varieties. Additional types of algebraic varieties, such as projective varieties, must be used also.

Let G be a connected linear algebraic group and B be a Borel subgroup of G. Then it can be shown, by a somewhat subtle argument, that B is its own normalizer in G. Now any maximal torus T of G, being a closed connected soluble subgroup, lies in some Borel subgroup B of G. Let $U = R_u(B)$ be the unipotent radical of B. Then we have

$$B = UT \quad \text{and} \quad U \cap T = 1.$$

Thus B is a semidirect product of the unipotent group U and the torus T.

For example, let $G = GL_n(k)$. Then G is a connected reductive group. The diagonal subgroup $T = D_n(k)$ is a maximal torus in G. The subgroup $B = T_n(k)$ of upper-triangular matrices is a Borel subgroup of G and the subgroup $U = U_n(k)$ of upper unitriangular matrices is the unipotent radical of B.

Let $N = N(T)$. Then it can be shown that

$$G = BNB$$

i.e. each double coset of B in G contains an element of N, so has form BnB for some $n \in N$. Since $W = N/T$ we have a natural homomorphism $\pi : N \to W$. Then we have, for $n, n' \in N$, $BnB = Bn'B$ if and only if $\pi(n) = \pi(n')$. This gives a bijective correspondence between the set $B\backslash G/B$ of double cosets of B in G and the set of elements of W. The factorisation $G = BNB$ is called the *Bruhat decomposition* of G.

It is a remarkable fact that the set of all subgroups of G containing B can be described in a simple way. Let $I = \{1, 2, \dots, n\}$ and $W = \langle s_1, \dots, s_n \rangle$ be a set of Coxeter generators for the Weyl group W. Let $J \subset I$ be any subset of I. Let W_J be the subgroup of W given by

$$W_J = \langle s_i \; ; \; i \in J \rangle .$$

Let N_J be the subgroup of N satisfying

$$N_J/T = W_J .$$

Then it can be shown that

$$P_J = BN_JB$$

is a subgroup of G containing B, and that any subgroup of G containing B is equal to P_J for some $J \subset I$. Furthermore $P_J = P_{J'}$ if and only if $J = J'$. We also have

$$\begin{aligned} \langle P_J , P_{J'} \rangle &= P_{J \cup J'} \\ P_J \cap P_{J'} &= P_{J \cap J'} \end{aligned}$$

thus there are 2^n subgroups of G containing B which form a lattice isomorphic to the lattice of subsets of the set I with n elements. A *parabolic subgroup* of G is a subgroup conjugate to P_J for some J. A subgroup of G is parabolic if and only if it contains a Borel subgroup of G. Each parabolic subgroup can be shown to be its own normalizer in G.

For example, let $G = GL_n(k)$. The subgroup $B = T_n(k)$ of upper triangular matrices in G is a Borel subgroup. The parabolic subgroups containing B are the 'staircase subgroups' of the form

$$P_J = \begin{pmatrix} \boxed{*} & & & * \\ & \boxed{*} & & \\ & & \boxed{*} & \\ 0 & & & \boxed{*} \end{pmatrix}$$

where the matrices in the diagonal blocks are non-singular and the entries in the blocks above the diagonal are arbitrary. The number of such staircase subgroups is 2^{n-1}, which relates to the fact that the Weyl group $W = S_n$ has $n-1$ Coxeter generators.

Returning to the general case let $U_J = R_u(P_J)$. Then there exists a closed subgroup L_J of P_J such that

$$P_J = U_JL_J , \qquad U_J \cap L_J = 1 .$$

L_J is called a Levi subgroup of P_J. L_J is a connected reductive group. Thus the parabolic subgroup P_J is a semidirect product of its unipotent radical with a Levi subgroup. Also any two Levi subgroups of P_J are conjugate in P_J.

For example in the parabolic subgroup P_J of $GL_n(k)$ given above we may take

$$U_J = \begin{pmatrix} \boxed{I} & & & * \\ & \boxed{I} & & \\ & & \boxed{I} & \\ 0 & & & \boxed{I} \end{pmatrix} \qquad L_J = \begin{pmatrix} \boxed{*} & & & 0 \\ & \boxed{*} & & \\ & & \boxed{*} & \\ 0 & & & \boxed{*} \end{pmatrix}.$$

We note that the Levi subgroup L_J is a connected reductive group containing T as a maximal torus whose Weyl group is

$$N_{L_J}(T)/T \cong W_J .$$

4 Roots and coroots

We shall now describe the root system and coroot system of a connected reductive group G. The root system lies in the character group of a maximal torus T of G and the coroot system lies in the cocharacter group of T, so we begin by defining the character and cocharacter groups of a torus.

Let T be a torus and $X = \mathrm{Hom}(T, k^*)$ be the set of homomorphisms of algebraic groups from T to the multiplicative group k^*. X has a natural structure of an additive group under the operation

$$(\chi_1 + \chi_2)t = \chi_1(t)\chi_2(t) \qquad \chi_1 , \chi_2 \in X , \ t \in T .$$

In the special case when $T \cong k^*$ we have

$$\mathrm{Hom}(k^* , k^*) \cong \mathbb{Z}$$

since the only algebraic group homomorphisms from k^* into itself are the maps $\lambda \to \lambda^m$ for $m \in \mathbb{Z}$. It follows that if

$$T \cong k^* \times \cdots \times k^* \qquad (n \text{ factors})$$

then

$$X \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \qquad (n \text{ factors}) .$$

X is called the character group of T.

Now let $Y = \mathrm{Hom}(k^*, T)$. Then we have

$$Y \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \qquad (n \text{ factors})$$

and Y is called the cocharacter group of T. The group operation on Y is given by

$$(\gamma_1 + \gamma_2)\lambda = \gamma_1(\lambda)\gamma_2(\lambda) \qquad \gamma_1, \gamma_2 \in Y , \ \lambda \in k^* .$$

There is a duality map

$$X \times Y \to \mathbb{Z}$$

relating the character and cocharacter groups of T. Given $\chi \in X$ and $\gamma \in Y$ we have $\chi \circ \gamma \in \mathrm{Hom}(k^*, k^*)$, thus $(\chi \circ \gamma)\lambda = \lambda^m$ for some $m \in \mathbb{Z}$. We write $\langle \chi, \gamma \rangle = m$ and then $\langle \chi, \gamma \rangle$ is the required duality map. It induces isomorphisms

$$X \cong \mathrm{Hom}(Y, \mathbb{Z}), \qquad Y \cong \mathrm{Hom}(X, \mathbb{Z}).$$

Now let T be a maximal torus in a connected reductive group G. Let $N = N(T)$ and $N/T = W$. Then W acts on T by conjugation, giving an action $t \to t^w$ for $t \in T$, $w \in W$. We may also define actions of W on X and Y by

$$\begin{aligned} {}^w\chi(t) &= \chi(t^w) \text{ for } \chi \in X, t \in T, w \in W \\ \gamma^w(\lambda) &= \gamma(\lambda)^w \text{ for } \gamma \in Y, \lambda \in k^*, w \in W \end{aligned}$$

Let B be a Borel subgroup of G containing T. It can be shown that there is a unique opposite Borel subgroup B^- such that $B \cap B^- = T$. We have $B = UT$, $B^- = U^-T$ where $U = R_u(B)$ and $U^- = R_u(B^-)$. The torus T acts on U and U^- by conjugation.

We now consider the minimal proper subgroups of U invariant under T. Each of these turns out to be isomorphic to the additive group k^+. T acts on each such subgroup by conjugation, thus giving a homomorphism

$$T \to \mathrm{Aut}\ k^+$$

where Aut k^+ is the group of all algebraic group automorphisms of k^+. Now the only algebraic group automorphisms of k^+ are the maps

$$\lambda \to \mu\lambda \qquad \mu \in k^*.$$

It follows that

$$\mathrm{Aut}\ k^+ \cong k^* \ .$$

Thus we obtain a homomorphism $T \to k^*$, i.e. an element of the character group X of T. In this way each minimal T-invariant subgroup of U gives an element $\alpha \in X$. Distinct subgroups of U give distinct elements of X. Let Φ^+ be the set of all elements of X arising in this way. Φ^+ is a finite subset of X called the set of positive roots. Each positive root $\alpha \in \Phi^+$ arises from a root subgroup $U_\alpha \subset U$.

Similarly we may consider the minimal T-invariant subgroups of U^-. These give a set of elements Φ^- in X called the set of negative roots. We have

$$\alpha \in \Phi^+ \quad \text{if and only if} \quad -\alpha \in \Phi^- .$$

The set $\Phi = \Phi^+ \cup \Phi^-$ is called the set of roots of G with respect to T.

We illustrate this idea in the example $G = GL_n(k)$. Let $T = D_n(k)$ be the diagonal subgroup and $B = T_n(k)$, $B^- = T_n^-(k)$ be the subgroups of upper triangular and lower triangular matrices in G. Then the positive root subgroups U_α, $\alpha \in \Phi^+$, have the form

$$\begin{pmatrix} 1 & & & \\ & \ddots & \lambda & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = I + \lambda E_{ij}$$

for fixed i, j with $i < j$, and where $\lambda \in k$ is in the (i,j)-position. The root α coming from this subgroup is the character

$$\begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \xrightarrow{\alpha} \lambda_i \lambda_j^{-1} \qquad i < j\,.$$

Similarly the negative root subgroups U_α, $\alpha \in \Phi^-$, have the form

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & \lambda & \ddots & \\ & & & 1 \end{pmatrix} = I + \lambda E_{ij}$$

for fixed i, j with $i > j$ and where $\lambda \in k$ is in the (i,j)-position. The root α coming from this subgroup is the character

$$\begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \xrightarrow{\alpha} \lambda_i \lambda_j^{-1} \qquad i > j\,.$$

We now return to the case of a general connected reductive group G. Let $\alpha \in \Phi$. Then $-\alpha \in \Phi$ also, and we have root subgroups U_α, $U_{-\alpha}$ of G. Consider the subgroup $\langle U_\alpha, U_{-\alpha}\rangle$ generated by $U_\alpha, U_{-\alpha}$. It can be shown that this subgroup is isomorphic to $SL_2(K)$ or $PSL_2(K) = SL_2(K)/Z$ and that there is a surjective homomorphism

$$SL_2(K) \xrightarrow{\phi} \langle U_\alpha, U_{-\alpha}\rangle$$

satisfying

$$\left\{\phi\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}\right\} = U_\alpha$$
$$\left\{\phi\begin{pmatrix}1 & 0 \\ * & 1\end{pmatrix}\right\} = U_{-\alpha}$$
$$\left\{\phi\begin{pmatrix}\lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix}\right\} = T\,.$$

Thus there is a homomorphism $k^* \xrightarrow{\alpha^v} T$ given by

$$\alpha^v(\lambda) = \phi\begin{pmatrix}\lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix}.$$

The element $\alpha^v \in \mathrm{Hom}(k^*, T) = Y$ is called the coroot of α. The root α and its coroot α^v are related by $\langle\alpha, \alpha^v\rangle = 2$. The set of all coroots α^v for $\alpha \in \Phi$ is called Φ^v. Φ^v is a finite subset of Y.

We have seen that the Weyl group W acts on both X and Y. We can now say more about this action. Let

$$n_\alpha = \phi\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} \in \langle U_\alpha, U_{-\alpha}\rangle\,.$$

Then n_α lies in $N(T)$, and since $W = N(T)/T$, the element n_α induces an element $s_\alpha \in W$. It can be shown that $s_\alpha = s_{-\alpha}$, $s_\alpha^2 = 1$ and that the set of all s_α for $\alpha \in \Phi$ generate the Weyl group W.

The element s_α acts on the character group X of T by

$$s_\alpha(\chi) = \chi - \langle\chi, \alpha^v\rangle\alpha \qquad \chi \in X$$

and on the cocharacter group Y of T by

$$s_\alpha(\gamma) = \gamma - \langle\alpha, \gamma\rangle\alpha^v \qquad \gamma \in Y\,.$$

Moreover $s_\alpha(\Phi) = \Phi$ and $s_\alpha(\Phi^v) = \Phi^v$ for all $\alpha \in \Phi$. Since the $s_\alpha, \alpha \in \Phi$, generate W it follows that

$$w(\Phi) = \Phi \qquad w(\Phi^v) = \Phi^v$$

for all $w \in W$. Thus the elements of the Weyl group permute the roots and also permute the coroots.

5 Classification of simple algebraic groups

Let G be a simple algebraic group over k, T be a maximal torus of G, and B a Borel subgroup of G containing T. Let Φ be the root system of G with