1 The elementary properties of groups

1.1 Definitions

All crystals and most molecules possess symmetry, which can be exploited to simplify the discussion of their physical properties. Changes from one configuration to an indistinguishable configuration are brought about by sets of symmetry operators, which form particular mathematical structures called *groups*. We thus commence our study of group theory with some definitions and properties of groups of abstract elements. All such definitions and properties then automatically apply to all sets that possess the properties of a group, including symmetry groups.

Binary composition in a set of abstract elements $\{g_i\}$, whatever its nature, is always written as a multiplication and is usually referred to as "multiplication" whatever it actually may be. For example, if g_i and g_j are operators then the product $g_i\ g_j$ means "carry out the operation implied by g_j and then that implied by g_i." If g_i and g_j are both n-dimensional square matrices then $g_i\ g_j$ is the matrix product of the two matrices g_i and g_j evaluated using the usual row × column law of matrix multiplication. (The properties of matrices that are made use of in this book are reviewed in Appendix A1.) Binary composition is *unique* but is not necessarily commutative: $g_i\ g_j$ may or may not be equal to $g_j\ g_i$. In order for a set of abstract elements $\{g_i\}$ to be a G, the law of binary composition must be defined and the set must possess the following four properties.

(i) *Closure*. For all g_i, with $g_j \in \{g_j\}$,

$$g_i\ g_j = g_k \in \{g_i\}, \quad g_k \text{ a unique element of } \{g_i\}. \tag{1}$$

Because g_k is a unique element of $\{g_i\}$, if each element of $\{g_i\}$ is multiplied from the left, or from the right, by a particular element g_j of $\{g_i\}$ then the set $\{g_i\}$ is regenerated with the elements (in general) re-ordered. This result is called the *rearrangement theorem*

$$g_j\ \{g_i\} = \{g_i\} = \{g_i\}\ g_j. \tag{2}$$

Note that $\{g_i\}$ means a set of elements of which g_i is a typical member, but in no particular order. The easiest way of keeping a record of the binary products of the elements of a group is to set up a *multiplication table* in which the entry at the intersection of the g_ith row and g_jth column is the binary product $g_i\ g_i = g_k$, as in Table 1.1. It follows from the rearrangement theorem that each row and each column of the multiplication table contains each element of G once and once only.

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Excerpt
More Information

Table 1.1. *Multiplication table for the group* G = $\{g_i\}$ *in which the product* $g_i\ g_j$ *happens to be* g_k.

G	g_i	g_j	g_k	...
g_i	${g_i}^2$	g_k	$g_i\ g_k$	
g_j	$g_j\ g_i$	g_j^2	$g_j\ g_k$	
g_k	$g_k\ g_i$	$g_k\ g_j$	g_k^2	
⋮				

(ii) Multiplication is *associative*. For all $g_i,\ g_j,\ g_k \in \{g_i\}$,

$$g_i(g_j\ g_k) = (g_i\ g_j)g_k. \tag{3}$$

(iii) The set $\{g_i\}$ contains the *identity* element E, with the property

$$E\ \ g_j = g_j\ \ E = g_j,\ \forall\ g_j \in \{g_i\}. \tag{4}$$

(iv) Each element g_i of $\{g_i\}$ has an *inverse* $g_i^{-1} \in \{g\}_i$ such that

$$g_i^{-1}\ \ g_i = g_i\ \ g_i^{-1} = E,\ g_i^{-1} \in \{g_i\},\ \forall\ g_i \in \{g_i\}. \tag{5}$$

The number of elements g in G is called the *order* of the group. Thus

$$\mathrm{G} = \{g_i\}, \quad i = 1, 2, \ldots, g. \tag{6}$$

When this is necessary, the order of G will be displayed in parentheses G(g), as in G(4) to indicate a group of order 4.

Exercise 1.1-1 With binary composition defined to be addition: (a) Does the set of positive integers $\{p\}$ form a group? (b) Do the positive integers p, including zero (0) form a group? (c) Do the positive (p) and negative ($-p$) integers, including zero, form a group? [*Hint*: Consider the properties (i)–(iv) above that must be satisfied for $\{g_i\}$ to form a group.]

The multiplication of group elements is not necessarily commutative, but if

$$g_i\ \ g_j = g_j\ \ g_i,\ \forall\ g_i, g_j \in \mathrm{G} \tag{7}$$

then the group G is said to be *Abelian*. Two groups that have the same multiplication table are said to be *isomorphous*. As we shall see, a number of other important properties of a group follow from its multiplication table. Consequently these properties are the same for isomorphous groups; generally it will be necessary to identify corresponding elements in the two groups that are isomorphous, in order to make use of the isomorphous property. A group G is finite if the number g of its elements is a finite number. Otherwise the group G is infinite, if the number of elements is denumerable, or it is continuous. The group of Exercise 1.1-1(c) is infinite. For finite groups, property (iv) is automatically fulfilled as a consequence of the other three.

If the sequence $g_i,\ g_i^2,\ g_i^3,\ \ldots$ starts to repeat itself at $g_i^{c+1} = g_i$, because $g_i^c = E$, then the set $\{g_i\ g_i^2\ g_i^3\ \ldots\ g_i^c = E\}$, which is the period of g_i, is a group called a *cyclic group*, C. The order of the cyclic group C is c.

Exercise 1.1-2 (a) Show that cyclic groups are Abelian. (b) Show that for a finite cyclic group the existence of the inverse of each element is guaranteed. (c) Show that $\omega = \exp(-2\pi i/n)$ generates a cyclic group of order n, when binary composition is defined to be the multiplication of complex numbers.

If every element of G can be expressed as a finite product of powers of the elements in a particular subset of G, then the elements of this subset are called the *group generators*. The choice of generators is not unique: generally, a minimal set is employed and the defining relations like $g_i = (g_j)^p\ (g_k)^q$, etc., where $\{g_j\ g_k\}$ are group generators, are stated. For example, cyclic groups are generated from just one element g_i.

Example 1.1-1 A permutation group is a group in which the elements are permutation operators. A permutation operator P rearranges a set of indistinguishable objects. For example, if

$$P\{a\ \ b\ \ c\ \ \ldots\} = \{b\ \ a\ \ c\ \ \ldots\} \tag{8}$$

then P is a particular permutation operator which interchanges the objects a and b. Since $\{a\ b\ \ldots\}$ is a set of indistinguishable objects (for example, electrons), the final configuration $\{b\ a\ c\ \ldots\}$ is indistinguishable from the initial configuration $\{a\ b\ c\ \ldots\}$ and P is a particular kind of symmetry operator. The best way to evaluate products of permutation operators is to write down the original configuration, thinking of the n indistinguishable objects as allocated to n boxes, each of which contains a single object only. Then write down in successive rows the results of the successive permutations, bearing in mind that a permutation other than the identity involves the replacement of the contents of two or more boxes. Thus, if P applied to the initial configuration means "interchange the contents of boxes i and j" (which initially contain the objects i and j, respectively) then P applied to some subsequent configuration means "interchange the contents of boxes i and j, whatever they currently happen to be." A number of examples are given in Table 1.2, and these should suffice to show how the multiplication table in Table 1.3 is derived. The reader should check some of the entries in the multiplication table (see Exercise 1.1-3).

The elements of the set $\{P_0\ P_1\ \ldots P_5\}$ are the permutation operators, and binary composition of two members of the set, say $P_3\ P_5$, means "carry out the permutation specified by P_5 and then that specified by P_3." For example, P_1 states "replace the contents of box 1 by that of box 3, the contents of box 2 by that of box 1, and the contents of box 3 by that of box 2." So when applying P_1 to the configuration $\{3\ 1\ 2\}$, which resulted from P_1 (in order to find the result of applying $P_1^2 = P_1\ P_1$ to the initial configuration) the contents of box 1 (currently 3) are replaced by those of box 3 (which happens currently to be 2 – see the line labeled P_1); the contents of box 2 are replaced by those of box 1 (that is, 3); and finally the contents of box 3 (currently 2) are replaced by those of box 2 (that is, 1). The resulting configuration $\{2\ 3\ 1\}$ is the same as that derived from the original configuration $\{1\ 2\ 3\}$ by P_2, and so

Table 1.2. *Definition of the six permutation operators of the permutation group* S(3) *and some examples of the evaluation of products of permutation operators.*

In each example, the initial configuration appears on the first line and the permutation operator and the result of the operation are on successive lines. In the last example, the equivalent single operator is given on the right.

The identity $P_0 = E$

	1	2	3	original configuration (which therefore labels the "boxes")
P_0	1	2	3	final configuration (in this case identical with the initial configuration)

The two cyclic permutations

	1	2	3		1	2	3
P_1	3	1	2	P_2	2	3	1

The three binary interchanges

	1	2	3		1	2	3		1	2	3
P_3	1	3	2	P_4	3	2	1	P_5	2	1	3

Binary products with P_1

	1	2	3	
P_1	3	1	2	P_1
$P_1\ P_1$	2	3	1	P_2
$P_2\ P_1$	1	2	3	P_0
$P_3\ P_1$	3	2	1	P_4
$P_4\ P_1$	2	1	3	P_5
$P_5\ P_1$	1	3	2	P_3

Table 1.3. *Multiplication table for the permutation group* S(3).

The box indicates the subgroup C(3).

S(3)	P_0	P_1	P_2	P_3	P_4	P_5
P_0	P_0	P_1	P_2	P_3	P_4	P_5
P_1	P_1	P_2	P_0	P_5	P_3	P_4
P_2	P_2	P_0	P_1	P_4	P_5	P_3
P_3	P_3	P_4	P_5	P_0	P_1	P_2
P_4	P_4	P_5	P_3	P_2	P_0	P_1
P_5	P_5	P_3	P_4	P_1	P_2	P_0

$$P_1\ P_1\{1\ 2\ 3\} = \{2\ 3\ 1\} = P_2\{1\ 2\ 3\} \tag{9}$$

so that $P_1\ P_1 = P_2$. Similarly, $P_2\ P_1 = P_0$, $P_3\ P_1 = P_4$, and so on. The equivalent single operators (products) are shown in the right-hand column in the example in the last part of Table 1.2. In this way, we build up the multiplication table of the group S(3), which is shown in Table 1.3. Notice that the rearrangement theorem (closure) is satisfied and that each element has an inverse. The set contains the identity P_0, and examples to demonstrate associativity are readily constructed (e.g. Exercise 1.1-4). Therefore this set of permutations is a group. The group of all permutations of N objects is called the symmetric group

S(N). Since the number of permutations of N objects is $N!$, the order of the symmetric group is $N!$, and so that of S(3) is $3! = 6$.

Exercise 1.1-3 Evaluate the products in the column headed P_3 in Table 1.3.

Exercise 1.1-4 (a) Using the multiplication table for S(3) in Table 1.3 show that $(P_3\ P_1)P_2 = P_3(P_1\ P_2)$. This is an example of the group property of associativity. (b) Find the inverse of P_2 and also the inverse of P_5.

Answers to Exercises 1.1

Exercise 1.1-1 (a) The set $\{p\}$ does not form a group because it does not contain the identity E. (b) The set $\{p \quad 0\}$ contains the identity 0, $p + 0 = p$, but the inverses $\{-p\}$ of the elements $\{p\}$, $p + (-p) = 0$, are not members of the set $\{p \quad 0\}$. (c) The set of positive and negative integers, including zero, $\{p\ \overline{p}\ 0\}$, does form a group since it has the four group properties: it satisfies closure, and associativity, it contains the identity (0), and each element p has an inverse $\overline{p}$ or $-p$.

Exercise 1.1-2 (a) $g_i^p\, g_i^q = g_i^{p+q} = g_i^{q+p} = g_i^q\, g_i^p$. (b) If $p < c$, $g_i^p\, g_i^{c-p} = g_i^c = E$. Therefore, the inverse of g_i^p is g_i^{c-p}. (c) $\omega^n = \exp(-2\pi \mathrm{i}) = 1 = E$; therefore $\{\omega\ \omega^2 \ldots \omega^n = E\}$ is a cyclic group of order n.

Exercise 1.1-3

P_0	1	2	3	
P_3	1	3	2	P_3
$P_1\ P_3$	2	1	3	P_5
$P_2\ P_3$	3	2	1	P_4
$P_3\ P_3$	1	2	3	P_0
$P_4\ P_3$	2	3	1	P_2
$P_5\ P_3$	3	1	2	P_1

Exercise 1.1-4 (a) From the multiplication table, $(P_3\ P_1)\ P_2 = P_4\ P_2 = P_3$ and $P_3\ (P_1\ P_2) = P_3\ P_0 = P_3$. (b) Again from the multiplication table, $P_2\ P_1 = P_0 = E$ and so $P_2^{-1} = P_1$; $P_5\ P_5 = P_0$, $P_5^{-1} = P_5$.

1.2 Conjugate elements and classes

If $g_i, g_j, g_k \in$ G and

$$g_i\ g_j\ g_i^{-1} = g_k \tag{1}$$

then g_k is the *transform* of g_j, and g_j and g_k are *conjugate* elements. A complete set of the elements conjugate to g_i form a *class*, $\mathscr{C}_i$. The number of elements in a class is called the *order* of the class; the order of $\mathscr{C}_i$ will be denoted by c_i.

Exercise 1.2-1 Show that E is always in a class by itself.

Example 1.2-1 Determine the classes of S(3). Note that $P_0 = E$ is in a class by itself; the class of E is always named $\mathscr{C}_1$. Using the multiplication table for S(3), we find

$$\begin{aligned}
P_0\,P_1\,P_0^{-1} &= P_1\,P_0 = P_1,\\
P_1\,P_1\,P_1^{-1} &= P_2\,P_2 = P_1,\\
P_2\,P_1\,P_2^{-1} &= P_0\,P_1 = P_1,\\
P_3\,P_1\,P_3^{-1} &= P_4\,P_3 = P_2,\\
P_4\,P_1\,P_4^{-1} &= P_5\,P_4 = P_2,\\
P_5\,P_1\,P_5^{-1} &= P_3\,P_5 = P_2.
\end{aligned}$$

Hence $\{P_1\ P_2\}$ form a class $\mathscr{C}_2$. The determination of $\mathscr{C}_3$ is left as an exercise.

Exercise 1.2-2 Show that there is a third class of S(3), $\mathscr{C}_3 = \{P_3\ P_4\ P_5\}$.

Answers to Exercises 1.2

Exercise 1.2-1 For any group G with $g_i \in$ G,

$$g_i\,E\,g_i^{-1} = g_i\,g_i^{-1} = E.$$

Since E is transformed into itself by every element of G, E is in a class by itself.

Exercise 1.2-2 The transforms of P_3 are

$$\begin{aligned}
P_0P_3P_0^{-1} &= P_3P_0 = P_3,\\
P_1P_3P_1^{-1} &= P_5P_2 = P_4,\\
P_2P_3P_2^{-1} &= P_4P_1 = P_5,\\
P_3P_3P_3^{-1} &= P_0P_3 = P_3,\\
P_4P_3P_4^{-1} &= P_2P_4 = P_5,\\
P_5P_3P_5^{-1} &= P_1P_5 = P_4.
\end{aligned}$$

Therefore $\{P_3\ P_4\ P_5\}$ form a class, $\mathscr{C}_3$, of S(3).

1.3 Subgroups and cosets

A subset H of G, H ⊂ G, that is itself a group with the same law of binary composition, is a *subgroup* of G. Any subset of G that satisfies closure will be a subgroup of G, since the other group properties are then automatically fulfilled. The region of the multiplication table of S(3) in Table 1.3 in a box shows that the subset $\{P_0\ P_1\ P_2\}$ is closed, so that this set is a subgroup of S(3). Moreover, since $P_1^2 = P_2,\ P_1^3 = P_1\ P_2 = P_0 = E$, it is a cyclic subgroup of order 3, C(3).

Given a group G with subgroup H ⊂ G, then g_r H, where $g_r \in$ G but $g_r \notin$ H unless g_r is $g_1 = E$, is called a *left coset* of H. Similarly, H g_r is a *right coset* of H. The $\{g_r\}$, $g_r \in$ G but $g_r \notin$ H, except for $g_1 = E$, are called *coset representatives*. It follows from the uniqueness of the product of two group elements (eq. (1.1.2)) that the elements of g_r H are distinct from those of g_s H when $s \neq r$, and therefore that

$$\mathrm{G} = \sum_{r=1}^{t} g_r\ \mathrm{H}, \quad g_r \in \mathrm{G},\ g_r \notin \mathrm{H}\ (\text{except for } g_1 = E),\ t = g/h, \tag{1}$$

where t is the *index* of H in G. Similarly, G may be written as the sum of t distinct right cosets,

$$\mathrm{G} = \sum_{r=1}^{t} \mathrm{H}\ g_r, \quad g_r \in \mathrm{G},\ g_r \notin \mathrm{H}\ (\text{except for } g_1 = E),\ t = g/h. \tag{2}$$

If H $g_r = g_r$ H, so that right and left cosets are equal for all r, then

$$g_r\ \mathrm{H}\ g_r^{-1} = \mathrm{H}\ g_r\ g_r^{-1} = \mathrm{H} \tag{3}$$

and H is transformed into itself by any element $g_r \in$ G that is not in H. But for any $h_j \in$ H

$$h_j\ \mathrm{H}\ h_j^{-1} = h_j\ \mathrm{H} = \mathrm{H} \qquad (\text{closure}). \tag{4}$$

Therefore, H is transformed into itself by all the elements of G; H is then said to be an *invariant* (*or normal*) *subgroup* of G.

Exercise 1.3-1 Prove that any subgroup of index 2 is an invariant subgroup.

Example 1.3-1 Find all the subgroups of S(3); what are their indices? Show explicitly which, if any, of the subgroups of S(3) are invariant.

The subgroups of S(3) are

$$\{P_0\ P_1\ P_2\} = \mathrm{C}(3),\ \{P_0\ P_3\} = \mathrm{H}_1,\ \{P_0\ P_4\} = \mathrm{H}_2,\ \{P_0\ P_5\} = \mathrm{H}_3.$$

Inspection of the multiplication table (Table 1.3) shows that all these subsets of S(3) are closed. Since $g = 6$, their indices t are 2, 3, 3, and 3, respectively. C(3) is a subgroup of S(3) of index 2, and so we know it to be invariant. Explicitly, a right coset expansion for S(3) is

$$\{P_0\ P_1\ P_2\} + \{P_0\ P_1\ P_2\}P_4 = \{P_0\ P_1\ P_2\ P_3\ P_4\ P_5\} = \mathrm{S}(3). \tag{5}$$

The corresponding left coset expansion with the same coset representative is

$$\{P_0\ P_1\ P_2\} + P_4\{P_0\ P_1\ P_2\} = \{P_0\ P_1\ P_2\ P_4\ P_5\ P_3\} = \mathrm{S}(3). \tag{6}$$

Note that the elements of G do not have to appear in exactly the same order in the left and right coset expansions. This will only be so if the coset representatives commute with every element of H. All that is necessary is that the two lists of elements evaluated from the coset expansions both contain each element of G once only. It should be clear from eqs. (5) and (6) that H $g_r = g_r$ H, where H = $\{P_0\ P_1\ P_2\}$ and g_r is P_4. An alternative way of testing for invariance is to evaluate the transforms of H. For example,

$$P_4\{C(3)\}P_4^{-1} = P_4\{P_0\ P_1\ P_2\}P_4^{-1} = \{P_4\ P_5\ P_3\}P_4 = \{P_0\ P_2\ P_1\} = C(3). \quad (7)$$

Similarly for P_3 and P_5, showing therefore that C(3) *is* an invariant subgroup of S(3).

Exercise 1.3-2 Show that C(3) is transformed into itself by P_3 and by P_5.

$H_1 = \{P_0\ P_3\}$ is not an invariant subgroup of S(3). Although

$$\{P_0\ P_3\} + \{P_0\ P_3\}P_1 + \{P_0\ P_3\}P_2 = \{P_0\ P_3\ P_1\ P_4\ P_2\ P_5\} = S(3), \quad (8)$$

showing that H_1 is a subgroup of S(3) of index 3,

$$\{P_0\ P_3\}P_1 = \{P_1\ P_4\},\ \text{but}\ P_1\{P_0\ P_3\} = \{P_1\ P_5\}, \quad (9)$$

so that right and left cosets of the representative P_1 are not equal. Similarly,

$$\{P_0\ P_3\}P_2 = \{P_2\ P_5\},\ \text{but}\ P_2\{P_0\ P_3\} = \{P_2\ P_4\}. \quad (10)$$

Consequently, H_1 is not an invariant subgroup. For H to be an invariant subgroup of G, right and left cosets must be equal for each coset representative in the expansion of G.

Exercise 1.3-3 Show that H_2 is not an invariant subgroup of S(3).

Answers to Exercises 1.3

Exercise 1.3-1 If $t = 2$, $G = H + g_2\,H = H + H\,g_2$. Therefore, $H\,g_2 = g_2\,H$ and the right and left cosets are equal. Consequently, H is an invariant subgroup.

Exercise 1.3-2 $P_3\{P_0\,P_1\,P_2\}P_3^{-1} = \{P_3\,P_4\,P_5\}P_3 = \{P_0\,P_2\,P_1\}$ and $P_5\{P_0\,P_1\,P_2\}P_5^{-1} = \{P_5\,P_3\,P_4\}P_5 = \{P_0\,P_2\,P_1\}$, confirming that C(3) *is* an invariant subgroup of S(3).

Exercise 1.3-3 A coset expansion for H_2 is

$$\{P_0\ P_4\} + \{P_0\ P_4\}P_1 + \{P_0\ P_4\}P_2 = \{P_0\ P_4\ P_1\ P_5\ P_2\ P_3\} = S(3).$$

The right coset for P_1 is $\{P_0\,P_4\}P_1 = \{P_1\ P_5\}$, while the left coset for P_1 is $P_1\ \{P_0\ P_4\} = \{P_1\ P_3\}$, which is not equal to the right coset for the same coset representative, P_1. So H_2 is not an invariant subgroup of S(3).

1.4 The factor group

Suppose that H is an invariant subgroup of G of index t. Then the t cosets g_r H of H (including g_1 H = H) each *considered as one element*, form a group of order t called the *factor group*,

$$\mathrm{F} = \mathrm{G}/\mathrm{H} = \sum_{r=1}^{t} (g_r\ \mathrm{H}),\ g_r \in \mathrm{G},\ g_r \notin \mathrm{H}\ (\text{except for } g_1 = E),\ t = g/h. \tag{1}$$

Each term in parentheses, g_r H, is *one* element of F. Because each element of F is a *set* of elements of G, binary composition of these sets needs to be defined. Binary composition of the elements of F is defined by

$$(g_p\ \mathrm{H})(g_q\ \mathrm{H}) = (g_p\ g_q)\ \mathrm{H}, \quad g_p, g_q \in \{g_r\}, \tag{2}$$

where the complete set $\{g_r\}$ contains $g_1 =$ E as well as the $t-1$ coset representatives that $\notin$ H. It follows from closure in G that $g_p\ g_q \in$ G. Because H is an invariant subgroup

$$g_r\ \mathrm{H} = \mathrm{H}\ g_r. \tag{3}$$

(2), (3) $$g_p\ \mathrm{H}\ g_q\ \mathrm{H} = g_p\ g_p\ \mathrm{H}\ \mathrm{H} = g_p\ g_q\ \mathrm{H}. \tag{4}$$

This means that in F

(4) $$\mathrm{H}\ \mathrm{H} = \mathrm{H}, \tag{5}$$

which is the necessary and sufficient condition for H to be the identity in F.

Exercise 1.4-1 Show that $g_1\ g_1 = g_1$ is both a necessary and sufficient condition for g_1 to be E, the identity element in G. [*Hint*: Recall that the identity element E is defined by

$$E\ g_i = g_i\ E = g_i,\ \forall\ g_i \in \mathrm{G}.] \tag{1.1.5}$$

Thus, F contains the identity: that {F} is indeed a group requires the demon-stration of the validity of the other group properties. These follow from the definition of binary composition in F, eq. (2), and the invariance of H in G.

Closure: To demonstrate closure we need to show that $g_p\ g_q$ H $\in$ F for $g_p, g_q, g_r \in \{g_r\}$. Now $g_p\ g_q \in$ G and so

(1) $$g_p\ g_q \in \{g_r\ \mathrm{H}\},\ r = 1, 2, \ldots, t, \tag{6}$$

(6) $$g_p\ g_q = g_r\ h_l,\ h_l \in \mathrm{H}, \tag{7}$$

(2), (7) $$g_p\ \mathrm{H}\ g_q\ \mathrm{H} = g_p\ g_q\ \mathrm{H} = g_r\ h_l\ \mathrm{H} = g_r\ \mathrm{H} \in \mathrm{F}. \tag{8}$$

Associativity:

(2), (3), (4) $$(g_p\ \mathrm{H}\ g_q\ \mathrm{H}) g_r\ \mathrm{H} = g_p\ g_q\ \mathrm{H}\ g_r\ \mathrm{H} = g_p\ g_q\ g_r\ \mathrm{H}, \tag{9}$$

(2), (3), (4) $$g_p\ \mathrm{H}(g_q\ \mathrm{H}\ g_r\ \mathrm{H}) = g_p\ H\ g_q\ g_r\ \mathrm{H} = g_p\ g_q\ g_r\ \mathrm{H}, \tag{10}$$

(9), (10) $$(g_p\ \mathrm{H}\ g_q\ \mathrm{H}) g_r\ \mathrm{H} = g_p\ \mathrm{H}(g_q\ \mathrm{H} g_r\ \mathrm{H}), \tag{11}$$

and so multiplication of the elements of {F} is associative.

Table 1.4. *Multiplication table of the factor group* $\mathrm{F} = \{E' \; P'\}$.

F	E'	P'
E'	E'	P'
P'	P'	E'

Inverse:

$$(2) \qquad (g_r^{-1}\ \mathrm{H})(g_r\ \mathrm{H}) = g_r^{-1}\ g_r\ \mathrm{H} = \mathrm{H}, \qquad (12)$$

so that the inverse of g_r H in F is g_r^{-1} H.

Example 1.4-1 The permutation group S(3) has the invariant subgroup $\mathrm{H} = \{P_0\ P_1\ P_2\}$. Here $g = 6$, $h = 3$, $t = 2$, and

$$\mathrm{G} = \mathrm{H} + P_3\ \mathrm{H},\ \ \mathrm{F} = \{\mathrm{H}\ P_3\ \mathrm{H}\} = \{E'\ P'\}, \qquad (13)$$

where the elements of F have primes to distinguish $\mathrm{E}' = \mathrm{H} \in \mathrm{F}$ from $E \in \mathrm{G}$.

$$(13), (2) \qquad P'P' = (P_3\ \mathrm{H})(P_3\ \mathrm{H}) = P_3\ P_3\ \mathrm{H} = P_0\ \mathrm{H} = \mathrm{H}. \qquad (14)$$

E' is the identity element in F, and so the multiplication table for the factor group of S(3), $\mathrm{F} = \{E'\ P'\}$, is as given in Table 1.4.

Exercise 1.4-2 Using the definitions of E' and P' in eq. (13), verify explicitly that $E'\ P' = P'$, $P'\ E' = P'$. [*Hint*: Use eq. (2).]

Exercise 1.4-3 Show that, with binary composition as multiplication, the set $\{1\ \ -1\ \ \mathrm{i}\ \ -\mathrm{i}\}$, where $\mathrm{i}^2 = -1$, form a group G. Find the factor group $\mathrm{F} = \mathrm{G}/\mathrm{H}$ and write down its multiplication table. Is F isomorphous with a permutation group?

Answers to Exercises 1.4

Exercise 1.4-1

$$(1.1.5) \qquad E\ E\ g_i = E\ g_i\ E = E\ g_i,\ \ \forall\ g_i \in \mathrm{G}, \qquad (15)$$

$$(15) \qquad E\ E = E, \qquad (16)$$

and so $E\ E = E$ is a *necessary* consequence of the definition of E in eq. (1.1.5). If $g_1\ g_1 = g_1$, then multiplying each side from the left or from the right by g_1^{-1} gives $g_1 = E$, which demonstrates that $g_1\ g_1 = g_1$ is a *sufficient* condition for g_1 to be E, the identity element in G.