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Preliminaries

1.1. Harmonic Mappings

A real-valued function $u(x, y)$ is *harmonic* if it satisfies Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A one-to-one mapping $u = u(x, y)$, $v = v(x, y)$ from a region D in the xy-plane to a region Ω in the uv-plane is a *harmonic mapping* if the two coordinate functions are harmonic. It is convenient to use the complex notation $z = x + iy$, $w = u + iv$ and to write $w = f(z) = u(z) + iv(z)$. Thus a complex-valued harmonic function is a harmonic mapping of a domain $D \subset \mathbb{C}$ if and only if it is *univalent* (or one-to-one) in D, that is, if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in D with $z_1 \neq z_2$. Here $\mathbb{C}$ denotes the complex plane.

It must be emphasized that in this book the term "harmonic mapping" will always mean a *univalent* complex-valued harmonic function, except for occasional discussion of higher-dimensional analogues. Some writers use the term in a broader sense that does not require univalence.

A complex-valued function $f = u + iv$ is *analytic* in a domain $D \subset \mathbb{C}$ if it has a derivative $f'(z)$ at each point $z \in D$. The *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are an immediate consequence. Conversely, if f has continuous first partial derivatives and the Cauchy–Riemann equations hold, then f is analytic in D. (See Ahlfors [3] for information about analytic functions.) It follows from the Cauchy–Riemann equations (and from the existence of higher derivatives) that every analytic function is harmonic. A pair of functions (u, v) that satisfy the Cauchy–Riemann equations is said to be a *conjugate pair*, and v is called the *harmonic conjugate* of u. Hence, $-u$ is the harmonic conjugate of v. Strictly speaking, the conjugate function is determined locally only up to an additive constant. In a multiply connected domain the conjugate function need not be single-valued.

An analytic univalent function is called a *conformal mapping* because it preserves angles between curves. In fact, this angle-preserving property characterizes analytic functions among all functions with continuous first partial derivatives and nonvanishing Jacobians, because it implies that the Cauchy–Riemann equations are satisfied.

The object of this book is to study complex-valued harmonic univalent functions whose real and imaginary parts are not necessarily conjugate. As soon as analyticity is abandoned, serious obstacles arise. Analytic functions are preserved under composition, but harmonic functions are not. A harmonic function of an analytic function is harmonic, but an analytic function of a harmonic function need not be harmonic. The analytic functions form an algebra, but the harmonic functions do not. Even the square or the reciprocal of a harmonic function need not be harmonic. The inverse of a harmonic mapping need not be harmonic. The boundary behavior of harmonic mappings may be much more complicated than that of conformal mappings. It will be seen, nevertheless, that much of the classical theory of conformal mappings can be carried over in some way to harmonic mappings.

The *Jacobian* of a function $f = u + iv$ is

$$J_f(z) = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x,$$

where the subscripts indicate partial derivatives. If f is analytic, its Jacobian takes the form $J_f(z) = (u_x)^2 + (v_x)^2 = |f'(z)|^2$. For analytic functions f, it is a classical result that $J_f(z) \neq 0$ if and only if f is locally univalent at z. Hans Lewy showed in 1936 that this remains true for harmonic mappings. A relatively simple proof will be given in Chapter 2. In view of Lewy's theorem, harmonic mappings are either *sense-preserving* (or *orientation-preserving*) with $J_f(z) > 0$, or *sense-reversing* with $J_f(z) < 0$ throughout the domain D where f is univalent. If f is sense-preserving, then $\overline{f}$ is sense-reversing. Conformal mappings are sense-preserving.

The simplest examples of harmonic mappings that need not be conformal are the *affine mappings* $f(z) = \alpha z + \gamma + \beta\overline{z}$ with $|\alpha| \neq |\beta|$. Affine mappings with $\gamma = 0$ are *linear mappings*. It is important to observe that every composition of a harmonic mapping with an affine mapping is again a harmonic mapping: if f is harmonic, then so is $\alpha f + \gamma + \beta\overline{f}$.

Another important example is the function $f(z) = z + \frac{1}{2}\overline{z}^2$, which maps the open unit disk $\mathbb{D}$ onto the region inside a hypocycloid of three cusps inscribed in the circle $|w| = \frac{3}{2}$. To verify its univalence, suppose $f(z_1) = f(z_2)$

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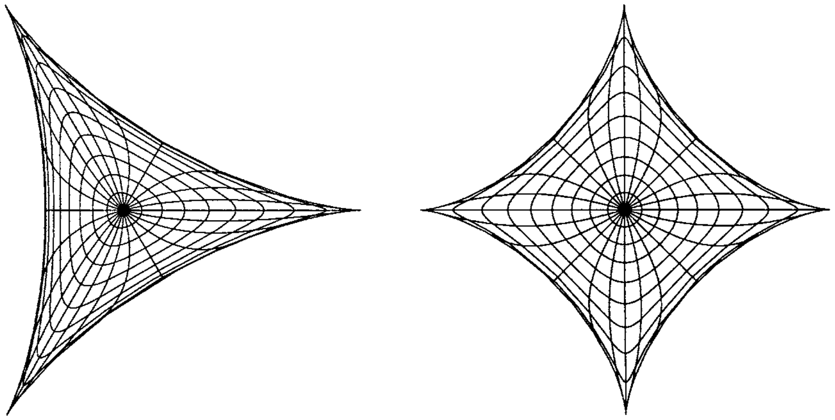

Figure 1.1. Image of mapping $f(z) = z + \frac{1}{n}\overline{z}^n$

for some points z_1 and z_2 in $\mathbb{D}$. Then

$$(\overline{z_1} + \overline{z_2})\,(\overline{z_1} - \overline{z_2}) = 2\,(z_2 - z_1).$$

But this is impossible unless $z_1 = z_2$, because $|z_1 + z_2| < 2$. The same argument shows that $f(z) = z + \frac{1}{n}\overline{z}^n$ is univalent for each $n \geq 2$.

The image of the disk under the mapping $f(z) = z + \frac{1}{n}\overline{z}^n$, as computed by *Mathematica*, is displayed graphically in Figure 1.1 for the cases $n = 2$ and 3. The curves in the figure are images of equally spaced concentric circles and radial segments. In general, the image of the disk under this mapping is bounded by a hypocycloid of $n + 1$ cusps inscribed in the circle $|w| = (n + 1)/n$.

In studying harmonic mappings of simply connected domains in the plane, there is no essential loss of generality in taking the unit disk as the domain of definition. To be more precise, suppose that f is a harmonic mapping of some simply connected domain $\Delta \subset \mathbb{C}$ onto a domain Ω, with $\Delta \neq \mathbb{C}$. The Riemann mapping theorem ensures the existence of a conformal mapping φ of $\mathbb{D}$ onto Δ. Thus the composition $F = f \circ \varphi$ is a harmonic mapping of $\mathbb{D}$ onto Ω. The original mapping is $f = F \circ \psi$, where ψ is the inverse of φ.

1.2. Some Basic Facts

Two simple differential operators appear commonly in complex analysis and are very convenient. They are

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

where $z = x + iy$. For a complex-valued function $f(z)$, the equation $\partial f/\partial\bar{z} = 0$ is just another way of writing the Cauchy–Riemann equations. A direct calculation shows that the Laplacian of f is

$$\Delta f = 4\frac{\partial^2 f}{\partial z \partial \bar{z}}.$$

Thus for functions f with continuous second partial derivatives, is is clear that f is harmonic if and only if $\partial f/\partial z$ is analytic. If f is analytic, then $\partial f/\partial z = f'(z)$, the ordinary derivative.

The operators $\partial/\partial z$ and $\partial/\partial\bar{z}$ are linear, and they have the usual properties of differential operators. For instance, the product and quotient rules hold:

$$\frac{\partial}{\partial z}(fg) = f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z},$$

$$\frac{\partial}{\partial z}\left(\frac{f}{g}\right) = g^{-2}\left(g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}\right),$$

and similarly for $\partial/\partial\bar{z}$. The special property

$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \bar{f}}{\partial \bar{z}}$$

connects the two derivatives. The differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

can be written as

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z},$$

thus motivating the notation $\partial/\partial z$ and $\partial/\partial\bar{z}$. The subscript notation $f_z = \partial f/\partial z$ and $f_{\bar{z}} = \partial f/\partial\bar{z}$ is often more convenient.

The chain rule for differentiation of composite functions can now be derived (formally). If $w = f(z)$ and $z = g(\zeta)$, then $w = h(\zeta)$, where $h = f \circ g$. Writing

$$dz = \frac{\partial g}{\partial \zeta}d\zeta + \frac{\partial g}{\partial \bar{\zeta}}d\bar{\zeta}$$

and

$$d\bar{z} = \frac{\partial \bar{g}}{\partial \zeta}d\zeta + \frac{\partial \bar{g}}{\partial \bar{\zeta}}d\bar{\zeta} = \overline{\frac{\partial g}{\partial \bar{\zeta}}}d\zeta + \overline{\frac{\partial g}{\partial \zeta}}d\bar{\zeta},$$

one finds after substitution that

$$dh = \frac{\partial f}{\partial z}\left(\frac{\partial g}{\partial \zeta}\,d\zeta + \frac{\partial g}{\partial \overline{\zeta}}\,d\overline{\zeta}\right) + \frac{\partial f}{\partial \overline{z}}\left(\overline{\frac{\partial g}{\partial \overline{\zeta}}}\,d\zeta + \overline{\frac{\partial g}{\partial \zeta}}\,d\overline{\zeta}\right).$$

Thus,

$$\frac{\partial h}{\partial \zeta} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial \zeta} + \frac{\partial f}{\partial \overline{z}}\overline{\frac{\partial g}{\partial \overline{\zeta}}} \qquad \text{and} \qquad \frac{\partial h}{\partial \overline{\zeta}} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial \overline{\zeta}} + \frac{\partial f}{\partial \overline{z}}\overline{\frac{\partial g}{\partial \zeta}}.$$

The Jacobian of a function $f = u + iv$ can be expressed as

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2.$$

Consequently, f is locally univalent and sense-preserving wherever $|f_z(z)| > |f_{\overline{z}}(z)|$, and sense-reversing where $|f_z(z)| < |f_{\overline{z}}(z)|$. Note that $f_z(z) \neq 0$ wherever $J_f(z) > 0$. For sense-preserving mappings $w = f(z)$ one sees that

$$(|f_z| - |f_{\overline{z}}|)|dz| \leq |dw| \leq (|f_z| + |f_{\overline{z}}|)|dz|.$$

These sharp inequalities have the geometric interpretation that f maps an infinitesimal circle onto an infinitesimal ellipse with

$$D_f = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}$$

as the ratio of the major and minor axes. The quantity $D_f = D_f(z)$ is called the *dilatation* of f at the point z. Clearly, $1 \leq D_f(z) < \infty$. A sense-preserving homeomorphism f is said to be *quasiconformal*, or *K-quasiconformal*, if $D_f(z) \leq K$ throughout the given region, where K is a constant and $1 \leq K < \infty$. The 1-quasiconformal mappings are simply the conformal mappings.

It is often more convenient to consider the ratio $\mu_f = f_{\overline{z}}/f_z$, called the *complex dilatation* of f. Thus, $0 \leq |\mu_f(z)| < 1$ if f is sense-preserving. It may be observed that $D_f(z) \leq K$ if and only if $|\mu_f(z)| \leq (K-1)/(K+1)$. It follows that a sense-preserving homeomorphism is quasiconformal if and only if its complex dilatation μ_f is bounded away from 1 in the given region: $|\mu_f(z)| \leq k < 1$. The mapping f is conformal if and only if $\mu_f = 0$. For the general theory of quasiconformal mappings the books by Lehto and Virtanen [1] and Ahlfors [1] are recommended.

In the theory of harmonic mappings, the quantity $\nu_f = \overline{f_{\overline{z}}}/f_z$, known as the *second complex dilatation*, turns out to be more relevant than the first complex dilatation μ_f. Since $|\nu_f| = |\mu_f|$, it is again clear that f is quasiconformal if and only if $|\nu_f(z)| \leq k < 1$.

Now let f be a complex-valued function defined in a domain $D \subset \mathbb{C}$ having continuous second partial derivatives. Suppose that f is locally univalent in D, with Jacobian $J_f(z) > 0$. Let $\omega = \nu_f = \overline{f_{\overline{z}}}/f_z$ be its second complex dilatation; then $|\omega(z)| < 1$ in D. Differentiating the equation $\overline{f_{\overline{z}}} = \omega f_z$ with respect to $\overline{z}$, one finds

$$\overline{f_{\overline{z}z}} = f_{z\overline{z}}\omega + f_z\omega_{\overline{z}}.$$

Now if f is harmonic in D, then $f_{z\overline{z}} = \frac{1}{4}\Delta f = 0$ there. Thus it follows that $\omega_{\overline{z}} = 0$ in D, so that ω is analytic. Conversely, if ω is analytic, then $\overline{f_{\overline{z}z}} = f_{z\overline{z}}\omega$. But since $|\omega(z)| < 1$, this implies that $f_{z\overline{z}} = 0$, and f is harmonic. Thus, f is harmonic if and only if ω is analytic. In particular, the second complex dilatation ω of a sense-preserving harmonic mapping f is always an analytic function of modulus less than one. This function ω will be called the *analytic dilatation* of f, or simply the dilatation when the context allows no confusion. Note that $\omega(z) \equiv 0$ if and only if f is analytic.

The analytic dilatation has some nice properties. For instance, if f is a sense-preserving harmonic mapping with analytic dilatation ω and it is followed by an affine mapping $A(w) = \alpha w + \gamma + \beta\overline{w}$ with $|\beta| < |\alpha|$, then the composition $F = A \circ f$ is a sense-preserving harmonic mapping with analytic dilatation

$$\frac{\overline{F_{\overline{z}}}}{F_z} = \frac{\overline{\alpha}\omega + \overline{\beta}}{\beta\omega + \alpha}.$$

For a proof, use the chain rule to calculate

$$F_z = A_w f_z + A_{\overline{w}}\overline{f_{\overline{z}}} = \alpha f_z + \beta\overline{f_{\overline{z}}},$$

$$F_{\overline{z}} = A_w f_{\overline{z}} + A_{\overline{w}}\overline{f_z} = \alpha f_{\overline{z}} + \beta\overline{f_z}.$$

Thus,

$$\frac{\overline{F_{\overline{z}}}}{F_z} = \frac{\overline{\alpha}\,\overline{f_{\overline{z}}} + \overline{\beta} f_z}{\beta\overline{f_{\overline{z}}} + \alpha f_z} = \frac{\overline{\alpha}\omega + \overline{\beta}}{\beta\omega + \alpha}.$$

The analytic dilatation also behaves well under precomposition. Let f be a sense-preserving harmonic mapping of a simply connected domain D onto a region Ω, with analytic dilatation ω. Let ψ map a domain Δ conformally onto D. Then the composition $F = f \circ \psi$ maps Δ harmonically onto Ω and has analytic dilatation $\omega \circ \psi$. To see this, simply use the chain rule to calculate $F_\zeta = f_z\psi'$ and $F_{\overline{\zeta}} = f_{\overline{z}}\overline{\psi'}$. Thus, the analytic dilatation of F is

$$\frac{\overline{F_{\overline{\zeta}}(\zeta)}}{F_\zeta(\zeta)} = \frac{\overline{f_{\overline{z}}(\psi(\zeta))}}{f_z(\psi(\zeta))} = \omega(\psi(\zeta)).$$

In a similar way, the first complex dilatation $\mu = f_{\overline{z}}/f_z$ shows a true invariance property. If f is followed by a conformal mapping φ and $F = \varphi \circ f$, then

F has the same complex dilatation μ. Indeed, the chain rule gives $F_z = \varphi' f_z$ and $F_{\bar{z}} = \varphi' f_{\bar{z}}$, so that $F_{\bar{z}}/F_z = f_{\bar{z}}/f_z$.

In a simply connected domain $D \subset \mathbb{C}$, a complex-valued harmonic function f has the representation $f = h + \overline{g}$, where h and g are analytic in D; this representation is unique up to an additive constant. For a proof, recall that f_z is analytic if f is harmonic, and let $h' = f_z$, where h is analytic in D. Now let $g = \overline{f} - \overline{h}$ and observe that

$$g_{\bar{z}} = \overline{f_z} - \overline{h_z} = 0 \qquad \text{in} \quad D$$

by the definition of h. Thus, g is analytic in D. The uniqueness of the representation depends on the fact that a function both analytic and anti-analytic must be constant. (An *anti-analytic function* is defined as the conjugate of an analytic function.) If f is real-valued, the representation reduces to $f = h + \overline{h} = \operatorname{Re}\{2h\}$, where $2h$ is the analytic completion of f, unique up to an additive imaginary constant. In a multiply connected domain, the representation $f = h + \overline{g}$ is valid locally but may not have a single-valued global extension.

For a harmonic mapping f of the unit disk $\mathbb{D}$, it is convenient to choose the additive constant so that $g(0) = 0$. The representation $f = h + \overline{g}$ is then unique and is called the *canonical representation* of f.

1.3. The Argument Principle

First recall the classical argument principle for analytic functions and its elegant proof. Let D be a domain bounded by a rectifiable Jordan curve C, oriented in the positive or "counterclockwise" direction. Let f be analytic in D and continuous in $\overline{D}$, with $f(z) \neq 0$ on C. The *index* or *winding number* of the image curve $f(C)$ about the origin is $I = (1/2\pi)\Delta_C \arg f(z)$, the total change in the argument of $f(z)$ as z runs once around C, divided by 2π. Let N be the total number of zeros of f in D, counted according to multiplicity. The argument principle asserts that $N = I$.

The customary proof begins with the observation that f'/f has a simple pole with residue n wherever f has a zero of order n, so the residue theorem gives

$$N = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\Delta_C \log f(z) = I.$$

(Actually, since the derivative $f'(z)$ need not be defined on C, the curve of integration should be slightly contracted.) As an application, it can be seen that if f is analytic in D and continuous in $\overline{D}$, and if it carries C in a sense-preserving manner onto a Jordan curve Γ bounding a domain Ω, then f maps D

univalently onto Ω. In other words, univalence on the boundary implies univalence in the interior.

Because the argument principle has so many important applications, it will be very useful to have a generalization to complex-valued harmonic functions. In fact, the theorem is essentially of topological nature and may be generalized in various ways to arbitrary continuous mappings. However, it is desirable both to avoid the complications of topological degree theory and to develop a precise extension of the argument principle to "sense-preserving" harmonic functions. The proof for analytic functions suggests that the structure of harmonic functions may allow an elementary approach to a more general form of the theorem, and this turns out to be the case.

A complex-valued harmonic function f, not identically constant, will be classified as sense-preserving in a domain D if it satisfies a Beltrami equation of the second kind, $\overline{f_{\overline{z}}} = \omega f_z$, where ω is an analytic function in D with $|w(z)| < 1$. Since the Jacobian is $J_f = |f_z|^2 - |f_{\overline{z}}|^2$, this implies in particular that $J_f(z) > 0$ wherever $f_z(z) \neq 0$. If $f(z_0) = 0$ at some point z_0 in D, the order of the zero can be defined in terms of the canonical decomposition $f = h + \overline{g}$. Write the power-series expansions of h and g as

$$h(z) = a_0 + \sum_{k=n}^{\infty} a_k (z - z_0)^k, \qquad g(z) = b_0 + \sum_{k=m}^{\infty} b_k (z - z_0)^k,$$

where $n \geq 1$, $m \geq 1$, and $a_n \neq 0$, $b_m \neq 0$. (Here it is tacitly assumed that f is not analytic.) Actually, $b_0 = -\overline{a_0}$ because $f(z_0) = 0$. The sense-preserving property of f takes the equivalent form $g' = \omega h'$, with $|\omega(z)| < 1$. From this it follows that $m > n$, or that $m = n$ and $|b_n| < |a_n|$. In either case, we will say that f has a zero of *order* n at z_0.

As an immediate consequence of the structural formula, it can be inferred that the zeros of a sense-preserving harmonic function are isolated. Indeed, if $f(z_0) = 0$, then for $0 < |z - z_0| < \delta$ it is possible to write

$$f(z) = h(z) + \overline{g(z)} = a_n (z - z_0)^n \{1 + \psi(z)\},$$

where

$$\psi(z) = (\overline{b_m}/\overline{a_n})(\overline{z} - \overline{z_0})^m (z - z_0)^{-n} + \cdots .$$

But it is clear that $|\psi(z)| < 1$ for z sufficiently close to z_0, since $m \geq n$ and $|b_n/a_n| < 1$ if $m = n$. Hence $f(z) \neq 0$ elsewhere near z_0, and the zeros of f are isolated. Observe that the sense-preserving hypothesis is essential, because the zeros of a harmonic function are not always isolated. For example, the function $f(z) = z + \overline{z} = 2x$ vanishes at every point on the imaginary axis.

The argument principle for harmonic functions can now be formulated as a direct generalization of the classical result for analytic functions.

Theorem. *Let f be a sense-preserving harmonic function in a Jordan domain D with boundary C. Suppose f is continuous in $\overline{D}$ and $f(z) \neq 0$ on C. Then $\Delta_c \arg f(z) = 2\pi N$, where N is the total number of zeros of f in D, counted according to multiplicity.*

Proof. Suppose first that f has no zeros in D, so that $N = 0$ and the origin lies outside $f(D \cup C)$. A fact from topology says that in this case $\Delta_c \arg f(z) = 0$, which proves the theorem. To prove the topological fact, let ϕ be a homeomorphism of the closed unit square S onto $D \cup C$ with $\phi : \partial S \to C$ a homeomorphism. Then the composition $F = f \circ \phi$ is a continuous mapping of S onto the plane with no zeros, and we want to prove that $\Delta_{\partial S} \arg F(z) = 0$. Begin by subdividing S into finitely many small squares S_j on each of which the argument of $F(z)$ varies by at most $\pi/2$. Then $\Delta_{\partial S_j} \arg F(z) = 0$ and so

$$\Delta_{\partial S} \arg F(z) = \sum_j \Delta_{\partial S_j} \arg F(z) = 0,$$

where the first equality relies on the cancellation of contributions from the ∂S_j except on ∂S.

Next suppose that f does have zeros in D. Because the zeros are isolated and f does not vanish on C, there are only a finite number of distinct zeros in D. Denote them by z_j for $j = 1, 2, \ldots, \nu$. Let γ_j be a circle of radius $\delta > 0$ centered at z_j, where δ is chosen so small that the circles γ_j all lie in D and do not meet each other. Join each circle γ_j to C by a Jordan arc λ_j in D. Consider the closed path Γ formed by moving around C in the positive direction while making a detour along each λ_j to γ_j, running once around this circle in the negative (clockwise) direction, then returning along λ_j to C. This curve Γ contains no zeros of f, and so $\Delta_\Gamma \arg f(z) = 0$ by the case just considered. But the contributions of the arcs λ_j along Γ cancel out, so that

$$\Delta_C \arg f(z) = \sum_{j=1}^{\nu} \Delta_{\gamma_j} \arg f(z),$$

where each of the circles γ_j is now traversed in the positive direction. This formula reduces the global problem to a local one. (The same reduction is often used to prove the residue theorem.)

Suppose now that f has a zero of order n at a point z_0. Then, as observed earlier, f has the local form

$$f(z) = a_n(z - z_0)^n\{1 + \psi(z)\}, \qquad a_n \neq 0,$$

where $|\psi(z)| < 1$ on a sufficiently small circle γ defined by $|z - z_0| = \delta$. This shows that

$$\Delta_\gamma \arg f(z) = n\Delta_\gamma \arg\{z - z_0\} + \Delta_\gamma \arg\{1 + \psi(z)\} = 2\pi n.$$

Therefore, if f has zeros of order n_j at the points z_j, the conclusion is that

$$\Delta_C \arg f(z) = \sum_{j=1}^{\nu} \Delta_{\gamma_j} \arg f(z) = 2\pi \sum_{j=1}^{\nu} n_j = 2\pi N,$$

which proves the theorem. The result admits an obvious extension to multiply connected domains, just as for analytic functions. ■

Several corollaries are worthy of note. First of all, there is a direct extension of Rouché's theorem to sense-preserving harmonic functions. Specifically, if p and $p + q$ are sense-preserving harmonic functions in D, continuous in $\overline{D}$, and $|q(z)| < |p(z)|$ on C, then p and $p + q$ have the same number of zeros inside D. As in the standard proof for analytic functions, the inequality on C implies that neither p nor $p + q$ has a zero on C and that the images of C under the two functions have the same winding numbers about the origin. Thus the harmonic version of Rouché's theorem follows from the harmonic version of the argument principle.

Next there is a generalization of Hurwitz's theorem. If f_n are harmonic functions in a domain D that converge locally uniformly, then their limit function f is harmonic. The harmonic version of Hurwitz's theorem asserts that if f and all of the f_n are sense-preserving, then a point z_0 in D is a zero of f if and only if it is a cluster point of zeros of the functions f_n. More precisely, f has a zero of order m at z_0 if and only if each small neighborhood of z_0 (small enough to contain no other zeros of f) contains precisely m zeros, counted according to multiplicity, of f_n for every n sufficiently large. The proof applies Rouché's theorem exactly as in the analytic case, with $p = f$ and $q = f_n - f$.

Finally, sense-preserving harmonic functions have the *open mapping property*: they carry open sets to open sets. In fact, as in the analytic case, a stronger statement can be made. If f is a sense-preserving harmonic function near a point z_0 where $f(z_0) = w_0$, and if $f(z) - w_0$ has a zero of order $n (n \geq 1)$ at z_0, then to each sufficiently small $\varepsilon > 0$ there corresponds a $\delta > 0$ with the following property. For each point $\alpha \in N_\delta(w_0) = \{w : |w - w_0| < \delta\}$, the function $f(z) - \alpha$ has exactly n zeros, counted according to multiplicity, in $N_\varepsilon(z_0)$. The proof appeals to the harmonic version of Rouché's theorem with $p = f - w_0$ and $q = w_0 - \alpha$.