Introduction to the theory of distributions

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with additional material by

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INTRODUCTION

The theory of distributions is a generalization of classical analysis, which makes it possible to deal in a systematic manner with difficulties which previously had been overcome by *ad hoc* constructions, or just by pure hand waving. In fact, it does a good deal more: it provides a new and wider framework, and a more perspicuous language, in which one can reformulate and develop classical problems. Its influence has been particularly pervasive and fruitful in the theory of linear partial differential equations.

Let us consider some examples. If $(x, t) \in \mathbf{R}^2$, then

$$u = f(x+t) + g(x-t)$$

satisfies d'Alembert's equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,$$

provided that the functions f and g are twice differentiable. This restriction is both irksome and unnatural in many instances. It can be overcome by introducing so-called weak solutions. By definition, these are functions u such that

$$\int u \left(\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} \right) \mathrm{d}x \, \mathrm{d}t = 0$$

for all sufficiently 'good' functions ϕ, for example for $\phi \in C_c^2(\mathbf{R}^2)$, the class of twice continuously differentiable functions that vanish on the exterior of a bounded set.

Again, if $x \in \mathbf{R}^3$, then the Newtonian potential

$$u(x) = \int \frac{f(y)}{|x-y|} \, \mathrm{d}y$$

satisfies Poisson's equation

$$\Delta u = -4\pi f$$

if the density function f is, for example, continuously differentiable. But it may not possess second order derivatives when f is merely continuous. Yet, it is then continuously differentiable and obeys Gauss's law that the flux of the field across a closed surface S is proportional to the matter enclosed by S. This difficulty can also be avoided by working with weak solutions of Poisson's equation. Furthermore, if one replaces f by the Dirac delta 'function', one obtains $u = 1/|x| = (x_1^2 + x_2^2 + x_3^2)^{-1/2}$ when $x \neq 0$, which is the potential due to a particle at the origin. It is suggestive to express this by writing

$$\Delta(1/|x|) = -4\pi\delta(x).$$

But this has to be interpreted; for example, one can take it to mean that

$$\int \frac{\Delta\phi(x)}{|x|}\,\mathrm{d}x = -4\pi\phi(0)$$

for all 'good' functions, say for all $\phi \in C_c^2(\mathbf{R}^3)$.

In all these cases, the difficulties and ambiguities disappear when the equations are read in terms of distributions. In the theory of distributions, functions are replaced by linear forms on an auxiliary vector space, whose members are called test functions. Recall that, if V is a vector space over the field $\mathbf{C}$ of complex numbers, then a linear form on V is a homomorphism $V \to \mathbf{C}$. The linear forms on V are made into a vector space $\mathrm{Hom}(V, \mathbf{C})$ in the obvious way: $\langle cu, \phi\rangle = c\langle u, \phi\rangle$ if $c \in \mathbf{C}$ and $u \in \mathrm{Hom}(V, \mathbf{C})$, and $\langle u + v, \phi\rangle = \langle u, \phi\rangle + \langle v, \phi\rangle$ if $u, v \in \mathrm{Hom}(V, \mathbf{C})$, where $\phi \in V$ in each case. In distribution theory, the basic space of test functions is $C_c^\infty(\mathbf{R}^n)$; its members are (complex valued) functions on $\mathbf{R}^n$ which possess continuous derivatives of all orders, and vanish outside some bounded set. The notations $C_0^\infty(\mathbf{R}^n)$, and L. Schwartz's original $\mathscr{D}(\mathbf{R}^n)$, are also used.

A continuous function f: $\mathbf{R}^n \to \mathbf{C}$ determines a linear form on $C_c^\infty(\mathbf{R}^n)$ by the rule

$$\langle f, \phi\rangle = \int_{\mathbf{R}^n} f\phi\,\mathrm{d}x, \quad \phi \in C_c^\infty(\mathbf{R}^n). \tag{1}$$

Conversely, it can (and will) be shown that this linear form determines f uniquely so that the space of continuous functions on $\mathbf{R}^n$ can be identified with a subspace of $\mathrm{Hom}(C_c^\infty(\mathbf{R}^n), \mathbf{C})$. If the function f is also continuously differentiable, then the linear forms on $C_c^\infty(\mathbf{R}^n)$ determined by its derivatives are, by (1) and an integration by parts,

$$\langle \partial f/\partial x_i, \phi\rangle = \int \phi(\partial f/\partial x_i)\,\mathrm{d}x = -\int f(\partial\phi/\partial x_i)\,\mathrm{d}x,$$

$$i = 1, \ldots, n,\ \phi \in C_c^\infty(\mathbf{R}^n).$$

Thus, for $i = 1, \ldots, n$,

$$\langle \partial f/\partial x_i, \phi\rangle = -\langle f, \partial\phi/\partial x_i\rangle, \quad \phi \in C_c^\infty(\mathbf{R}^n). \tag{2}$$

But this makes sense for any linear form on $C_c^\infty(\mathbf{R}^n)$, and so provides a definition of the derivatives of such a form. As one can iterate (2), one thus obtains (generalized) derivatives of all orders.

Multiplication by smooth (infinitely differentiable) functions can also be defined by analogy with the special case (1); one simply puts

$$\langle fu, \phi\rangle = \langle u, f\phi\rangle, \quad \phi \in C_c^\infty(\mathbf{R}^n). \tag{3}$$

By combining (2) and (3), one can thus account for the action of any linear differential operator with smooth coefficients on $\operatorname{Hom}(C_c^\infty(\mathbf{R}^n), \mathbf{C})$.

There is one other essential ingredient. The class of distributions is not the whole of $\operatorname{Hom}(C_c^\infty(\mathbf{R}^n), \mathbf{C})$: it is the subspace consisting of continuous linear forms. To say this, presupposes that $C_c^\infty(\mathbf{R}^n)$ has been equipped with an appropriate topology. The choice of this topology is, in fact, a cardinal feature of the theory of distributions. To define it, and to explore its implications, one must appeal to the theory of locally convex topological vector spaces. However, the course adopted in this book is to specify a certain set of inequalities which a linear form on $C_c^\infty(\mathbf{R}^n)$ must satisfy in order to qualify as a distribution. Once these are granted, the theory can be built up systematically and logically. But, so as to give some idea of what is involved, to readers who either do not have the time, or the inclination, to go into this more fully, a sketch of the functional-analytic background is given as an Appendix at the end of the book. This can be omitted, and the reader who does so should also ignore references to topological vector spaces in the text.